Lecture Notes On

Mathematics for Business, Economics and Social Sciences

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Chapter 1

Functions

Equations: We can perfom the same operation on both sides of an equality:

$$\begin{aligned} 8x-2 &= 5x+7 && \text{Add 2 to both sides} \\ 8x &= 5x+9 && \text{Subtract } 5x \text{ from both sides} \\ 3x &= 9 && \text{Multiply both sides by } \tfrac{1}{3} \\ x &= 3 \end{aligned}$$

Example 1–1: Solve the equation $\dfrac{2x}{2x+5} = \dfrac{3}{4}$

Solution:

$$\begin{aligned} \frac{2x}{2x+5} &= \frac{3}{4} \\ 8x &= 6x+15 \\ 2x &= 15 \\ x &= \frac{15}{2} \end{aligned}$$

Example 1–2: Solve the equation $\left|3x-12\right| = 27$

Solution: Using the definition of absolute value, we get

$$\begin{aligned} 3x-12 &= 27 \quad \text{or} \quad & -3x+12 &= 27 \\ 3x &= 39 \quad \text{or} \quad & -3x &= 15 \\ x &= 13 \quad \text{or} \quad & x &= -5 \end{aligned}$$

Intervals:

- Closed interval: $[a,\, b] = \{x : a \leqslant x \leqslant b\}$
- Open interval: $(a,\, b) = \{x : a < x < b\}$
- Half-open interval: $(a,\, b] = \{x : a < x \leqslant b\}$
- Unbounded interval: $(a,\, \infty) = \{x : a < x\}$

We will use $\mathbb{R}$ to denote all real numbers, in other words the interval $(-\infty,\, \infty)$.

Inequalities: Inequalities are similar to equations. We can add the same quantity to both sides, but if we multiply by a negative number, the direction of the inequality is reversed.

Example 1–3: Solve the inequality $7x - 5 \leqslant 30$.

Solution:

$$\begin{aligned} 7x - 5 &\leqslant 23 \\ 7x &\leqslant 28 \\ x &\leqslant 4 \\ x &\in \big(-\infty, 4\big] \end{aligned}$$

Example 1–4: Solve the inequality $\big|x + 10\big| < 11$.

Solution:

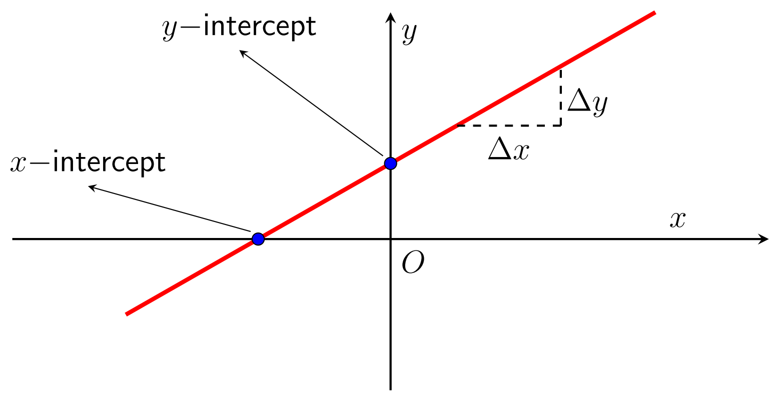
$$\begin{array}{rcccl} -11 & < & x+10 & < & 11 \\ -21 & < & x & < & 1 \end{array}$$
$$x \in \big(-21, 1\big)$$

Example 1–5: Solve the inequality $\big|x + 10\big| > 11$.

Solution:

$$\begin{array}{rclcrcl} x+10 & > & 11 & \text{or} & x+10 & < & -11 \\ x & > & 1 & \text{or} & x & < & -21 \\ x & \in & \big(-\infty, -21\big) & \text{or} & x & \in & \big(1, \infty\big) \end{array}$$

Lines on the Plane:



Slope of a line is:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$x = 0$ gives the $y-$intercept and $y = 0$ gives the $x-$intercept.

- Slope-intercept equation: $y = mx + n$.
- Point-slope equation: $y - y_1 = m(x - x_1)$.

If we are given two points on a line or one point and the slope, we can find the equation of the line.

- If two lines are parallel: $m_1 = m_2$.
- If two lines are perpendicular: $m_1 \cdot m_2 = -1$.

The equation $x = c$ gives a vertical line and $y = c$ gives a horizontal line.

Example 1–6: Find the equation of the line passing through the points $(2, 9)$ and $(4, 13)$.

Solution: Let's find the slope first: $m = \dfrac{13-9}{4-2} = 2$

Now, let's use the point-slope form of a line equation using the point $(2, 9)$:

$$(y-9) = 2(x-2)$$

$$y = 2x+5$$

If we use $(4, 13)$, we will obtain the same result:

$$(y-13) = 2(x-4)$$

$$y = 2x+5$$

Example 1–7: Find the equation of the line passing through the point $(2, 4)$ and parallel to the line $3x+5y=1$.

Solution: If we rewrite the line equation as: $y = -\dfrac{3}{5}x + \dfrac{1}{5}$

we see that $m = -\dfrac{3}{5}$.

Therefore: $y - 4 = -\dfrac{3}{5}(x-2)$

$$y = -\frac{3}{5}x + \frac{26}{5}$$

or $3x+5y=26$.

Example 1–8: Find the equation of the line passing through the points $(24, 0)$ and $(8, -6)$.

Solution: The slope is:

$$m = \frac{-6-0}{8-24} = \frac{-6}{-16} = \frac{3}{8}$$

Using point-slope equation, we find:

$$y - 0 = \frac{3}{8}(x-24) \quad \Rightarrow \quad y = \frac{3}{8}x - 9$$

In other words: $3x - 8y = 72$.

Example 1–9: Find the equation of the line passing through origin and parallel to the line $2y - 8x - 12 = 0$.

Solution: If we rewrite the line equation as:

$$y = 4x+6,$$

we see that $m = 4$. Therefore:

$$y - 0 = 4(x-0)$$

$$y = 4x.$$

Note that a line through origin has zero intercept.

Function: A function f defined on a set D of real numbers is a rule that assigns to each number x in D exactly one real number, denoted by $f(x)$.
For example:

$$\begin{aligned} f(x) &= x^2 \\ f(x) &= 7x+2 \\ f(x) &= |x| \\ f(x) &= \frac{1}{x} \end{aligned}$$

Domain: The set D of all numbers for which $f(x)$ is defined is called the domain of the function f.

For example, consider the function $f(x) = 4x^2 + 5$.

There's no x value where $f(x)$ is undefined so its domain is $\mathbb{R}$.

Range: The set of all values of $f(x)$ is called the range of f.

$$\begin{aligned} x^2 &\geqslant 0 \\ 4x^2 &\geqslant 0 \\ 4x^2+5 &\geqslant 5 \end{aligned}$$

So range of $f(x) = 4x^2 + 5$ is: $\big[5, \infty\big)$.

The range and domain of a linear function $f(x) = ax + b$ is $\mathbb{R}$.

Example 1–10: Find the domain of the function

$$f(x) = \frac{1}{x+8}$$

Solution: Division by zero is undefined, in other words, it is not possible to evaluate this function at the point $x = -8$.

Therefore the domain is: $\mathbb{R} \setminus \{-8\}$.

We can also write this as: $\big(-\infty, -8\big) \cup \big(-8, \infty\big)$

Example 1–11: Find the domain of the function

$$f(x) = \sqrt{x-4}$$

Solution: Square root of a negative number is undefined. (In this course, we are not using complex numbers.) Therefore

$$x - 4 \geqslant 0 \quad \Rightarrow \quad x \geqslant 4$$

In other words, domain is: $\big[4, \infty\big)$

Example 1–12: Find the domain of the function

$$f(x) = \frac{1}{\sqrt{x-4}}$$

Solution: This is similar to previous exercise, but the function is not defined at $x = 4$. Therefore, the domain is: $\big(4, \infty\big)$

EXERCISES

Perform the following operations. Transform and simplify the result.

1–1) $(2^3)^2$

1–2) $\left(\dfrac{1}{16}\right)^{3/4}$

1–3) $72^{1/2}$

1–4) $\sqrt[3]{-125}$

1–5) $\sqrt[3]{\dfrac{8}{1000}}$

1–6) $\sqrt{\dfrac{48}{49}}$

1–7) $(a+b)^2$

1–8) $(a+b)(a-b)$

1–9) $\dfrac{1}{\sqrt{5}-\sqrt{3}}$

1–10) $\dfrac{12}{\sqrt{7}-1}-\dfrac{12}{\sqrt{7}+1}$

Perform the following operations. Transform and simplify the result.

1–11) $\sqrt{x^2\sqrt{x}}$

1–12) $\sqrt{x^3y}\,\sqrt{64xy^9}$

1–13) x^3-1

1–14) $\left(\sqrt{x^2+4}+3\right)\left(\sqrt{x^2+4}-3\right)$

1–15) x^4-100y^4

1–16) $\left(\dfrac{x^2\,y^{1/2}}{x^{2/3}\,y^{1/6}}\right)^3$

1–17) $(3a-2b)^2$

1–18) $(a+b)^3$

1–19) $\dfrac{2x}{x^2-4}+\dfrac{5}{x+2}$

1–20) $1-\dfrac{1}{1+\dfrac{1}{x}}$

Solve the following equations and inequalities:

1–21) $3(x+7) - 2(3x-4) = 14$

1–22) $\dfrac{x}{3} - \dfrac{x}{5} = \dfrac{7}{30}$

1–23) $\sqrt{x^2+16} = 5$

1–24) $|x-2| = 12$

1–25) $\big|x-7\big| < 8$

1–26) $\big|2x+6\big| \leqslant 4$

1–27) $\big|5x-10\big| > 15$

1–28) $\big|12-7x\big| \geqslant 1$

1–29) $\big|x^2-5\big| < 2$

1–30) $\big|x^2-5\big| < 10$

Find the equations of the following lines:

1–31) Passes through origin and has slope $m = \dfrac{1}{5}$.

1–32) Passes through the point $(-2, 6)$ and has slope $m = 3$.

1–33) Passes through the points $(-8, 2)$ and $(-1, -2)$.

1–34) Passes through $(0, -3)$ and parallel to the line $10y - 5x = 99$.

1–35) Passes through $(9, 12)$ and perpendicular to the line $2x + 5y = 60$.

Find the domain and range of the following functions:

1–36) $f(x) = \sqrt{10-x}$

1–37) $f(x) = x^2 + 12x + 35$

1–38) $f(x) = 8x - x^2$

1–39) $f(x) = \dfrac{1}{x^2 - 6x + 9}$

1–40) $f(x) = \dfrac{3}{x-7}$

ANSWERS

1–1) $2^3 \cdot 2^3 = 2^6 = 64$

1–2) $\left(2^{-4}\right)^{3/4} = 2^{-3} = \frac{1}{8}$

1–3) $\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$

1–4) $\left[(-5)^3\right]^{1/3} = -5$

1–5) $\frac{\sqrt[3]{8}}{\sqrt[3]{1000}} = \frac{2}{10} = 0.2$

1–6) $\frac{4\sqrt{3}}{7}$

1–7) $a^2 + 2ab + b^2$

1–8) $a^2 - b^2$

1–9) $\frac{\sqrt{5}+\sqrt{3}}{2}$

1–10) 4

1–11) $x^{5/4}$

1–12) $8x^2y^5$

1–13) $(x-1)(x^2+x+1)$

1–14) $x^2 - 5$

1–15) $(x^2 - 10y^2)(x^2 + 10y^2)$

1–16) x^4y

1–17) $9a^2 - 12ab + 4b^2$

1–18) $a^3 + 3a^2b + 3ab^2 + b^3$

1–19) $\frac{7x-10}{x^2-4}$

1–20) $\frac{1}{x+1}$

1–21) $x = 5$

1–22) $x = \dfrac{7}{4}$

1–23) $x = \pm 3$

1–24) $x = 14$ or $x = -10$

1–25) $-1 < x < 15$

1–26) $-5 \leqslant x \leqslant -1$

1–27) $x < -1$ or $x > 5$

1–28) $x \leqslant \dfrac{11}{7}$ or $x \geqslant \dfrac{13}{7}$

1–29) $\sqrt{3} < x < \sqrt{7}$ or $-\sqrt{7} < x < -\sqrt{3}$

1–30) $-\sqrt{15} < x < \sqrt{15}$

1–31) $y = \dfrac{1}{5}x$

1–32) $y = 3x + 12$

1–33) $4x + 7y + 18 = 0$

1–34) $x - 2y = 6$

1–35) $5x - 2y = 21$

1–36) Domain: $\big(-\infty, 10\big]$, range: $\big[0, \infty\big)$.

1–37) Domain: $\mathbb{R}$, range: $\big[-1, \infty\big)$.

1–38) Domain: $\mathbb{R}$, range: $\big(-\infty, 16\big)$.

1–39) Domain: $\mathbb{R} \setminus \{3\}$, range: $\big(0, \infty\big)$.

1–40) Domain: $\mathbb{R} \setminus \{7\}$, range: $\big(-\infty, 0\big) \cup \big(0, \infty\big)$.

Chapter 2

Parabolas

Quadratic Equations: The solution of the equation

$$ax^2 + bx + c = 0$$

is:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} \quad \text{where} \quad \Delta = b^2 - 4ac$$

Here, we assume $a \neq 0$.

- If $\Delta > 0$, there are two distinct solutions.
- If $\Delta = 0$, there is a single solution.
- If $\Delta < 0$, there is no real solution.

(In this course, we only consider real numbers)

Example 2–1: Solve the equation $x^2 - 6x - 7 = 0$.

Solution: We can factor this equation as: $(x - 7)(x + 1) = 0$

Therefore $x - 7 = 0$ or $x + 1 = 0$.

In other words, $x = 7$ or $x = -1$.

Alternatively, we can use the formula to obtain the same result. Note that

$$a = 1, \; b = -6 \quad \text{and} \quad c = -7$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{6 \pm \sqrt{36 + 28}}{2} \\ &= \frac{6 \pm 8}{2} \end{aligned}$$

So $x = 7$ or $x = -1$.

Example 2–2: Solve $8x^2 - 6x - 5 = 0$.

Solution: Using the formula, we obtain:

$$x = \frac{6 \pm \sqrt{36 + 160}}{16} = \frac{6 \pm 14}{16}$$

So $x = \frac{5}{4}$ or $x = -\frac{1}{2}$.

Alternatively, we can see directly that

$(4x - 5)(2x + 1) = 0$, but this is not easy.

Example 2–3: Solve $9x^2 - 12x + 4 = 0$.

Solution: If we can see that this is a full square

$(3x - 2)^2 = 0$ we obtain $x = \frac{2}{3}$ easily.

Alternatively, $\Delta = (-12)^2 - 4 \cdot 9 \cdot 4 = 0$

(There is only one solution)

Example 2–4: Solve $3x^2 + 6x + 4 = 0$.

Solution:
$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= 36 - 48 \\ &= -12 \end{aligned}$$

$\Delta < 0 \quad \Rightarrow \quad$ There is no solution.

Quadratic Functions:

A function of the form

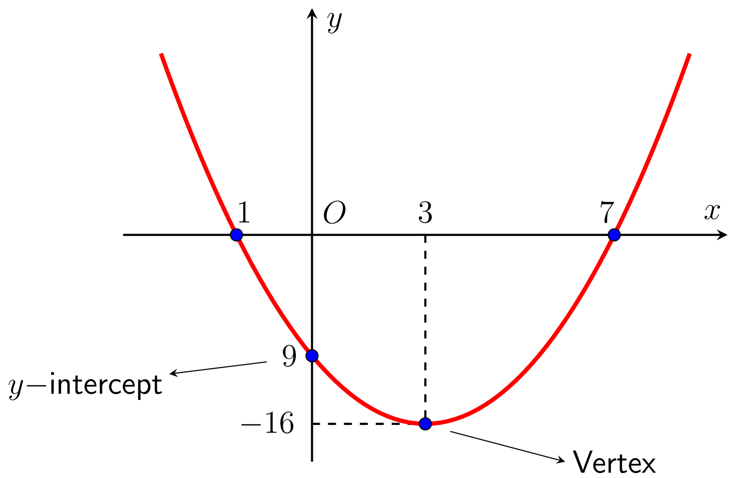
$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

is called a quadratic function. The graph of a quadratic function is a parabola.

- If $a > 0$, the arms of the parabola open upward.
- If $a < 0$, the arms of the parabola open downward.

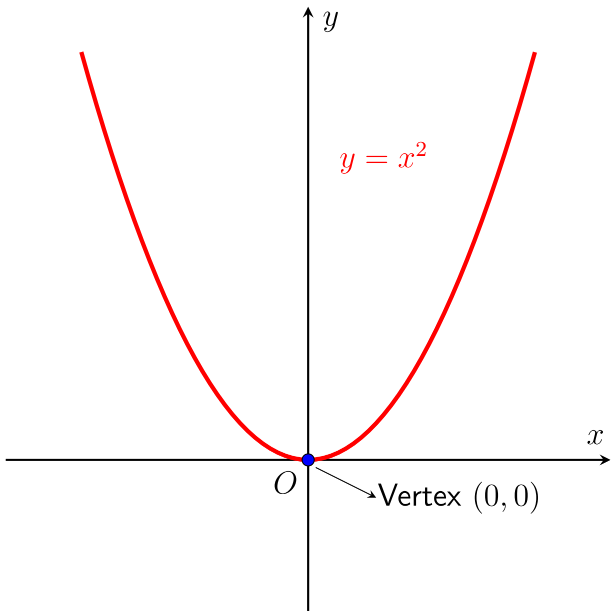
The vertex of the parabola is maximum or minimum point.

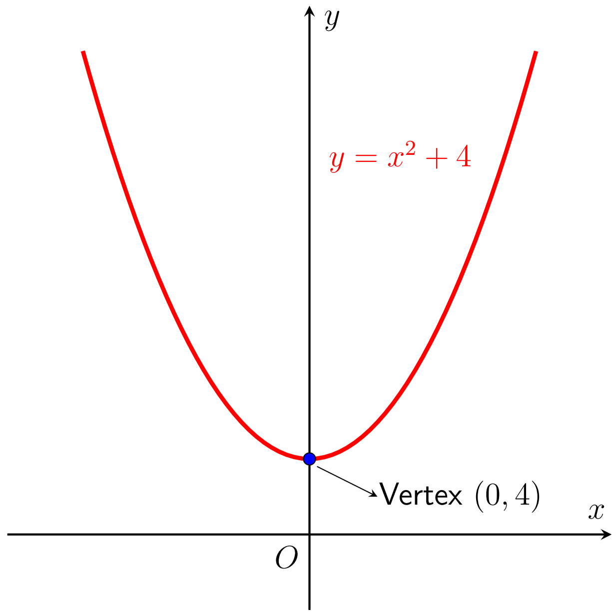
The $x-$coordinate of the vertex is $-\frac{b}{2a}$ and the $y-$coordinate is $f\left(-\frac{b}{2a}\right)$. An example is:

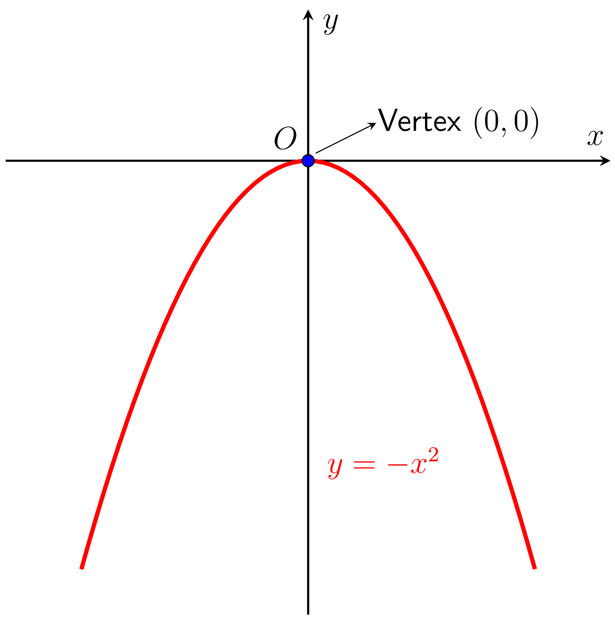


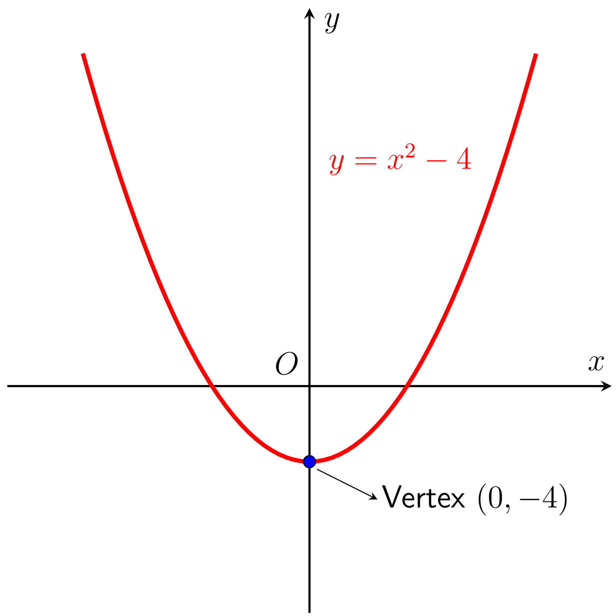
The graph of $y = x^2 - 6x - 7$

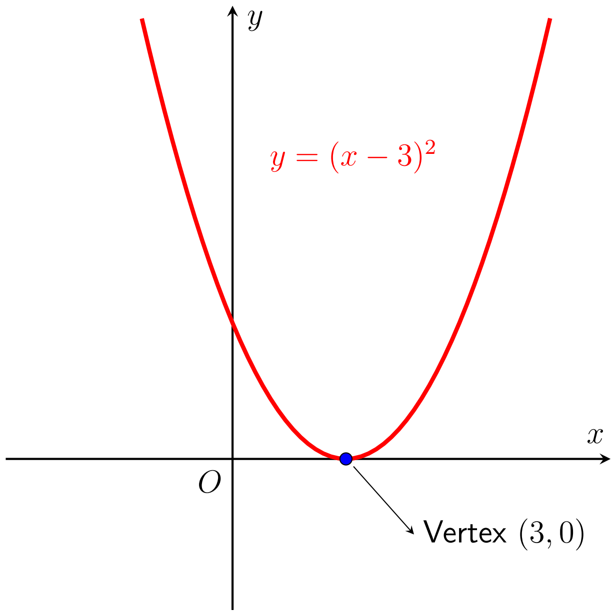
Examples of some parabolas and their vertices:

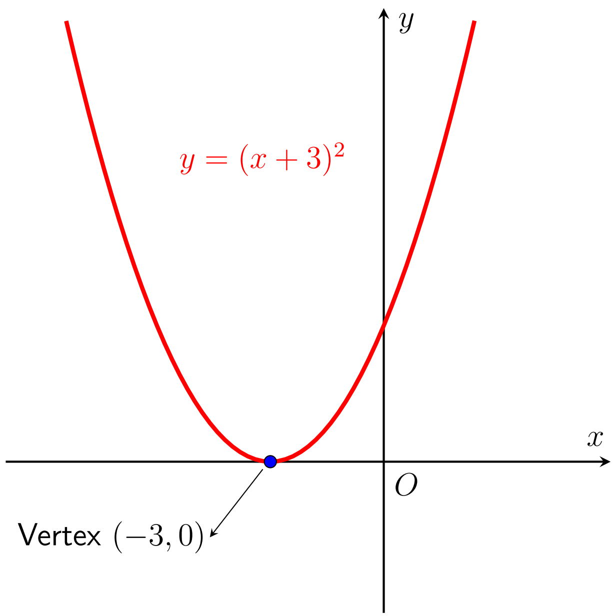












Example 2–5: Sketch the graph of $f(x) = x^2 - 10x + 16$.

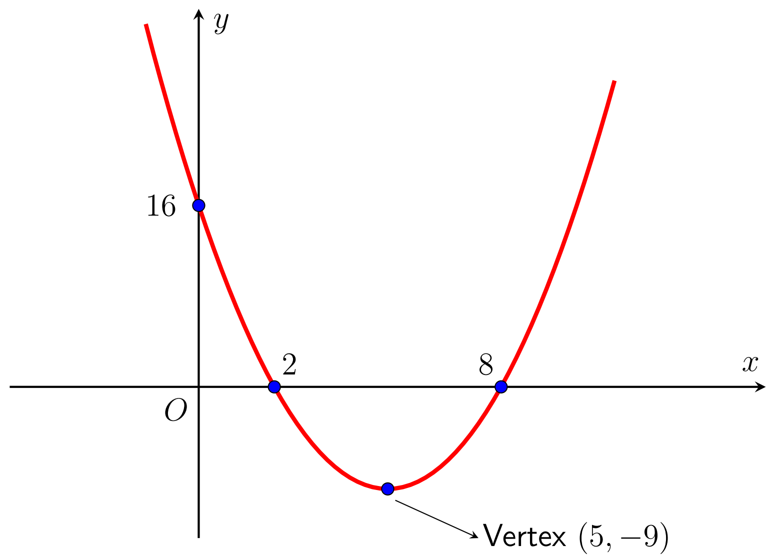
Solution: y–intercept: $x = 0 \quad \Rightarrow \quad y = 16$

Roots: $x^2 - 10x + 16 = 0 \quad \Rightarrow \quad x = 2$ or $x = 8$.

Vertex: $-\dfrac{b}{2a} = \dfrac{10}{2} = 5, \quad f(5) = -9.$

The coordinates of the vertex is $(5, -9)$.

$a > 0 \quad \Rightarrow \quad$ arms open upward. The graph is:



We can obtain the same graph by writing the given function in the form:

$$f(x) = (x-5)^2 - 9$$

EXERCISES

Solve the following quadratic equations:

2–1) $x^2 - 5x - 24 = 0$

2–2) $2x^2 + 9x - 5 = 0$

2–3) $6x^2 - 7x + 2 = 0$

2–4) $49x^2 - 14x + 1 = 0$

2–5) $4x^2 + 6x + 3 = 0$

2–6) $x^2 - 17x = 0$

2–7) $4x^2 - 20x + 25 = 0$

2–8) $x^2 - 4x + 5 = 0$

2–9) $x^2 - \dfrac{10}{3}x + 1 = 0$

2–10) $x^2 - 2x - 1 = 0$

Find the vertex and $x-$ and $y-$ intercepts of the following parabolas. Sketch their graphs:

2–11) $y = x^2 - 6x$

2–12) $y = -x^2 + 12$

2–13) $y = x^2 - 4x - 21$

2–14) $y = -x^2 + 3x + 4$

2–15) $y = x^2 + 10x + 25$

2–16) $y = 4x^2 - 8x + 3$

2–17) $y = 5x^2 + 15$

2–18) $y = -(x - 4)^2$

2–19) $y = x^2 - 4x + 5$

2–20) $y = -3x^2 + 60x - 450$

ANSWERS

2–1) $x_1 = 8, \quad x_2 = -3.$

2–2) $x_1 = \frac{1}{2}, \quad x_2 = -5.$

2–3) $x_1 = \frac{1}{2}, \quad x_2 = \frac{2}{3}.$

2–4) $x_1 = \frac{1}{7}$. (double root.)

2–5) There is no solution.

2–6) $x_1 = 0, \quad x_2 = 17.$

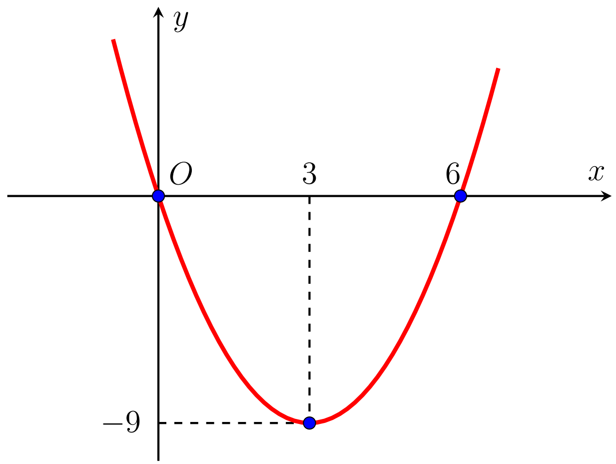
2–7) $x_1 = \frac{5}{2}$. (double root.)

2–8) There is no solution.

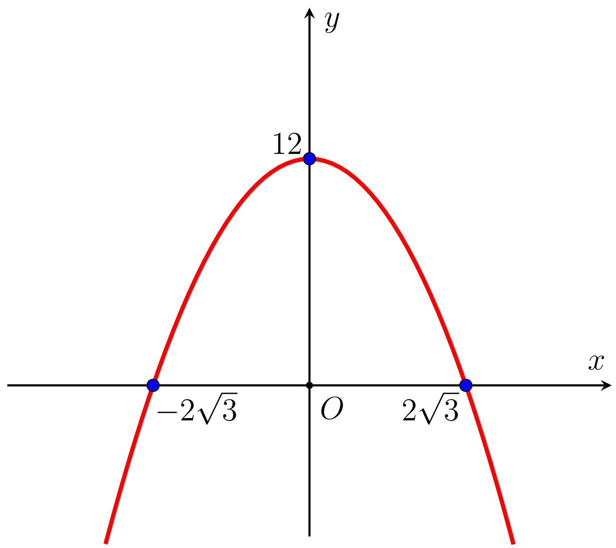
2–9) $x_1 = 3, \quad x_2 = \frac{1}{3}.$

2–10) $x_1 = 1 + \sqrt{2}, \quad x_2 = 1 - \sqrt{2}.$

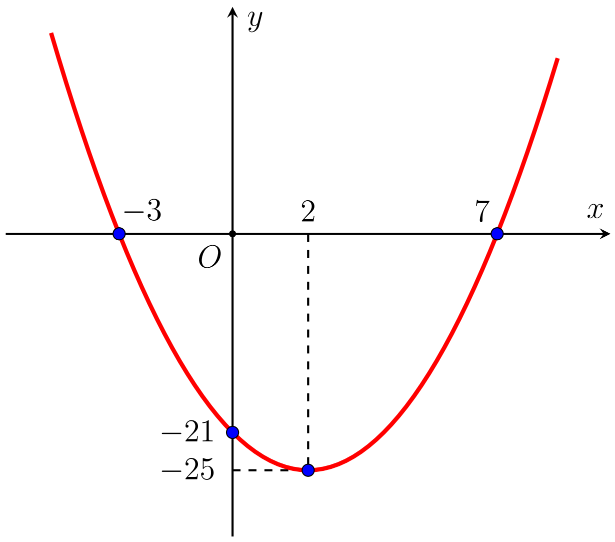
2–11)



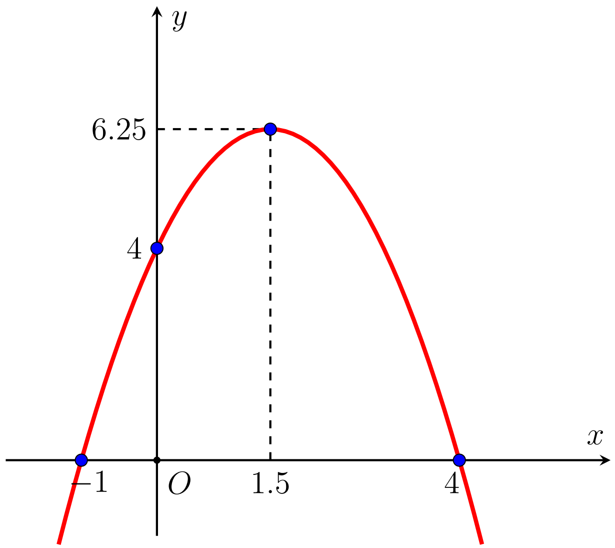
2–12)



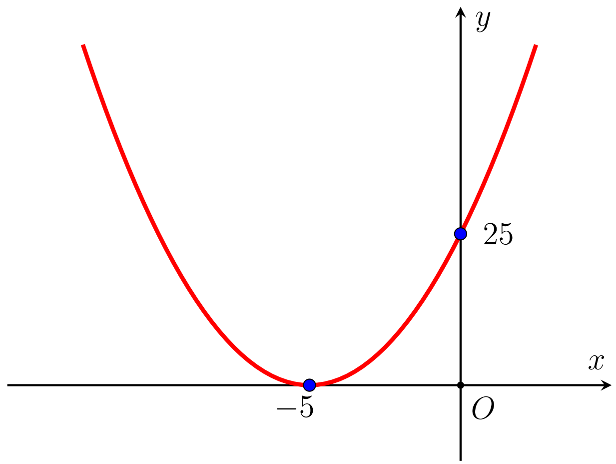
2–13)



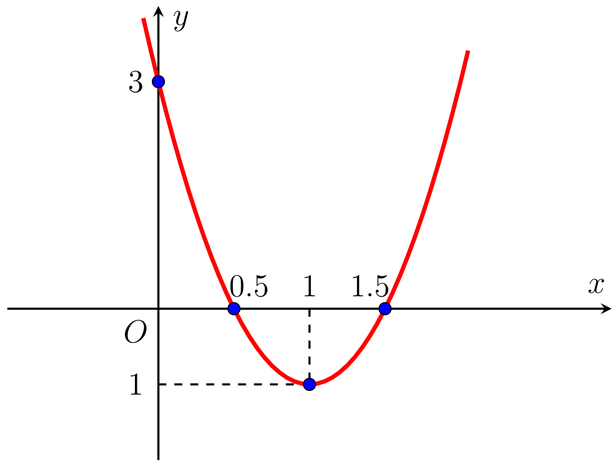
2–14)



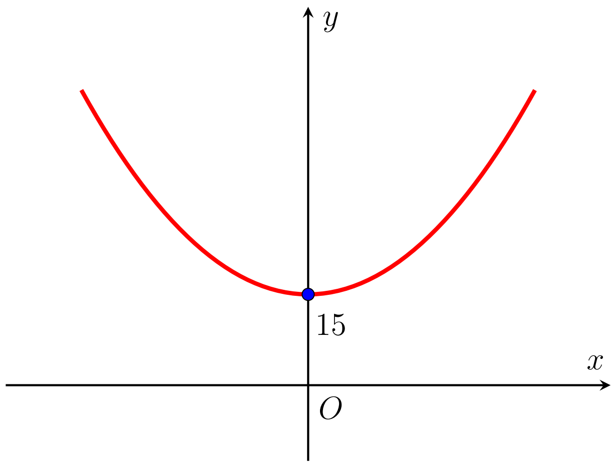
2–15)



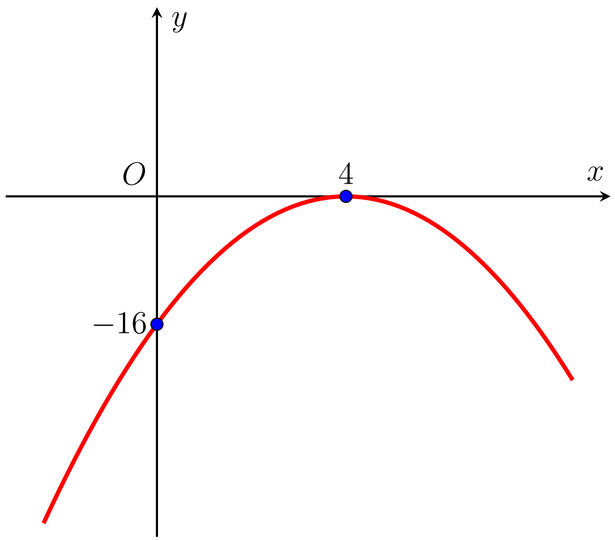
2–16)



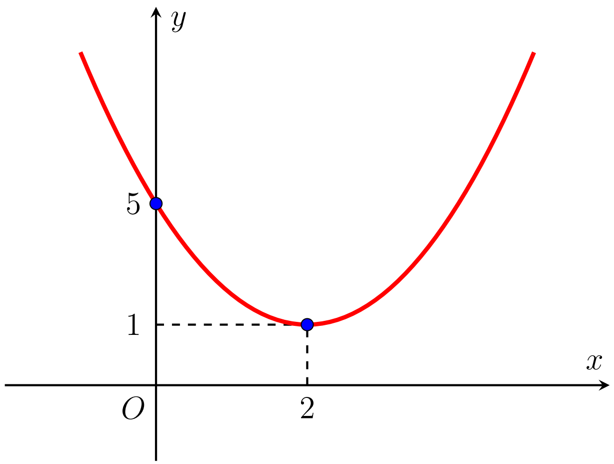
2–17)



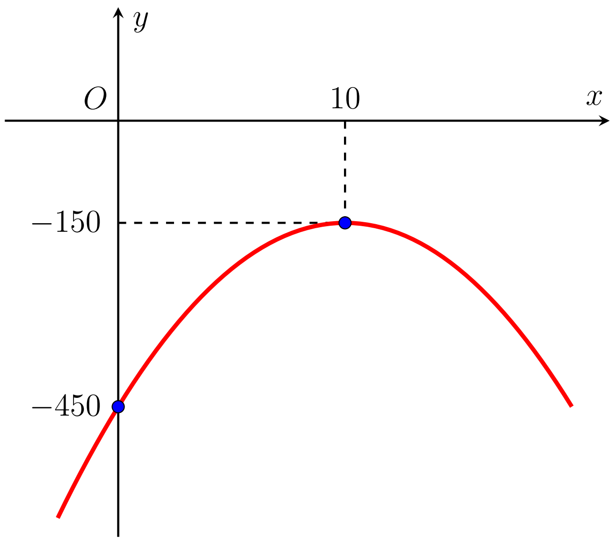
2–18)



2–19)



2–20)



Chapter 3

Exponential and Logarithmic Functions

Polynomials: A function of the form

$$p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a polynomial of degree n. For example, $120x^5 - 17x + \dfrac{7}{2}$ is a polynomial.

$\sqrt{x}$, x^{-1}, $\dfrac{1}{1+x}$, $x^{5/3}$ are NOT polynomials.

Rational Functions: The quotient of two polynomials is a rational function $f(x) = \dfrac{p(x)}{q(x)}$. For example,

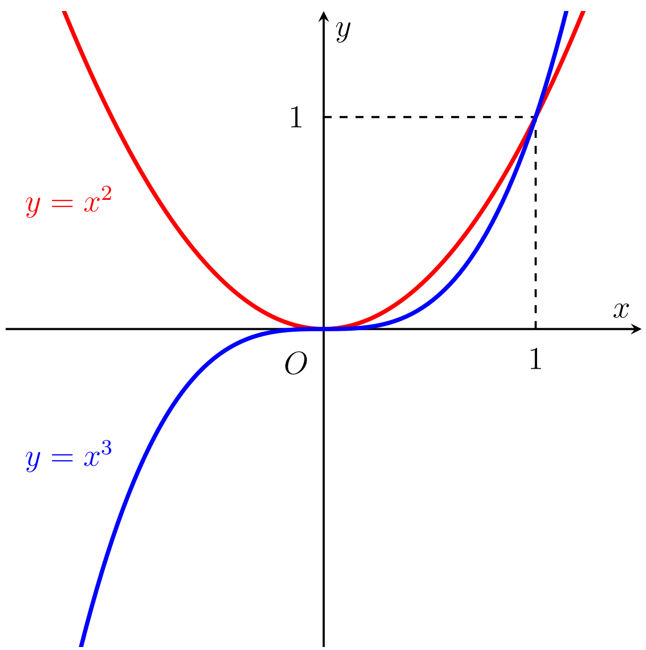
$$\frac{3x^2 - 5}{1 + 2x - 7x^3}$$

is a rational function.

Question: What is the domain of a polynomial? A rational function?

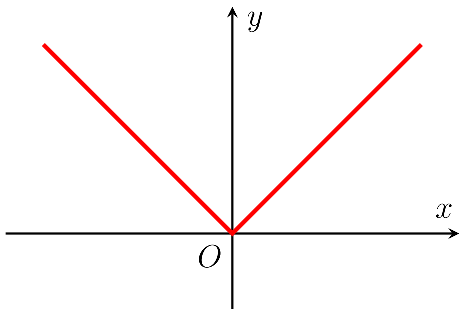
Example 3–1: Sketch the functions $y = x^2$ and $y = x^3$ on the same coordinate system.

Solution:

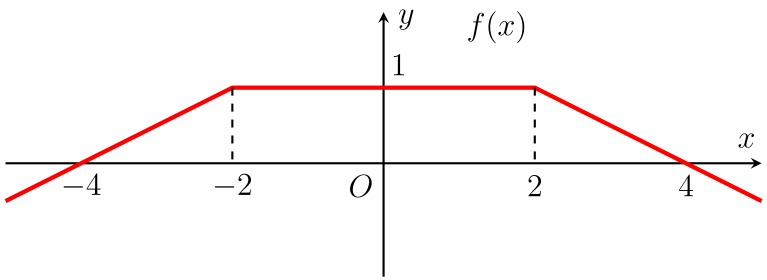


Piecewise-Defined Functions: We may define a function using different formulas for different parts of the domain. For example, the absolute value function is:

$$|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$$



Example 3–2: Find the formula of the function $f(x)$:



Solution:

$$f(x) = \begin{cases} \dfrac{x+4}{2} & \text{if} \quad x < -2 \\ 1 & \text{if} \quad -2 \leqslant x \leqslant 2 \\ \dfrac{-x+4}{2} & \text{if} \quad x > 2 \end{cases}$$

Inverse Functions:
If $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$, the functions f and g are inverses of each other.

For example, the inverse of $f(x) = 2x + 1$ is:

$$f^{-1}(x) = g(x) = \frac{x-1}{2}$$

One–to–one Functions: If $f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$ then f is one-to one.

For example, $f(x) = x^3$ is one-to one but $g(x) = x^2$ is not, because $g(1) = g(-1)$.

Onto Functions: Let $f : A \to B$. If there exists an $x \in A$ for all $y \in B$ such that $f(x) = y$ then f is onto.

For example, $f(x) = 2x + 1$ is onto but $g(x) = |x|$ is not, because there is no x such that $g(x) = -2$ or any other negative number.

Theorem: A function has an inverse if and only if it is one-to-one and onto.

Example 3–3: Find the inverse of the function $f(x) = \dfrac{x-2}{x+1}$ on the domain $\mathbb{R} \setminus \{-1\}$ and range $\mathbb{R} \setminus \{1\}$.

Solution: $y = \dfrac{x-2}{x+1} \quad \Rightarrow \quad yx + y = x - 2$

$$yx - x = -y - 2 \quad \Rightarrow \quad x(y-1) = -y - 2$$

$$x = -\frac{y+2}{y-1}$$

In other words, $f^{-1}(x) = -\dfrac{x+2}{x-1}$.

Exponential Functions: Functions of the form

$$f(x) = a^x$$

where a is a positive constant (but $a \neq 1$) are called exponential functions. The domain is:

$$\mathbb{R} = (-\infty, \infty)$$

and the range is

$$(0, \infty)$$

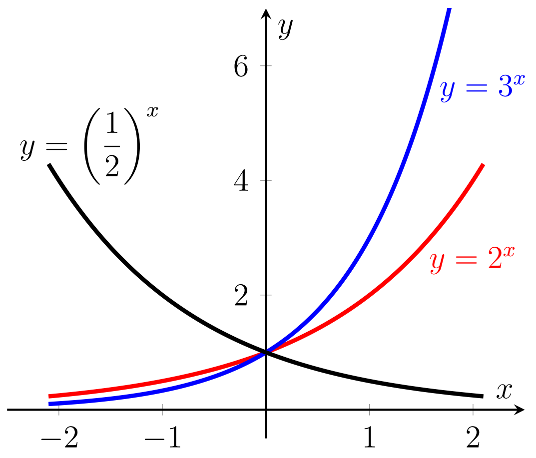
Remember that:

- $a^n = a \cdot a \cdots a$
- $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$
- $a^{1/n} = \sqrt[n]{a}$
- $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

The natural exponential function is:

$$f(x) = e^x$$

where $e = 2.71828\ldots$.



For the exponential function $f(x) = 2^x$,

$$\begin{aligned} f(6) &= 64 \\ f(5) &= 32 \\ f(1) &= 2 \\ f(0) &= 1 \\ f\left(\frac{1}{2}\right) &= \sqrt{2} \end{aligned}$$

Do not confuse this with the polynomial function $g(x) = x^2$ because

$$\begin{aligned} g(6) &= 36 \\ g(5) &= 25 \\ g(1) &= 1 \\ g(0) &= 0 \\ g\left(\frac{1}{2}\right) &= \frac{1}{4} \end{aligned}$$

Example 3–4: If we invest an amount A in the bank, and if the rate of interest is 15% per year, how much money will we have after n years?

Solution: We are multiplying by 1.15 every year, so: $1.15^n A$.

Example 3–5: A firm has C customers now. Every month, 30% of the customers leave. How many remain after n months?

Solution: We are multiplying by 0.7 every month, so: $0.7^n C$.

Logarithmic Functions: The inverse of the exponential function $y = a^x$ is the logarithmic function with base a:

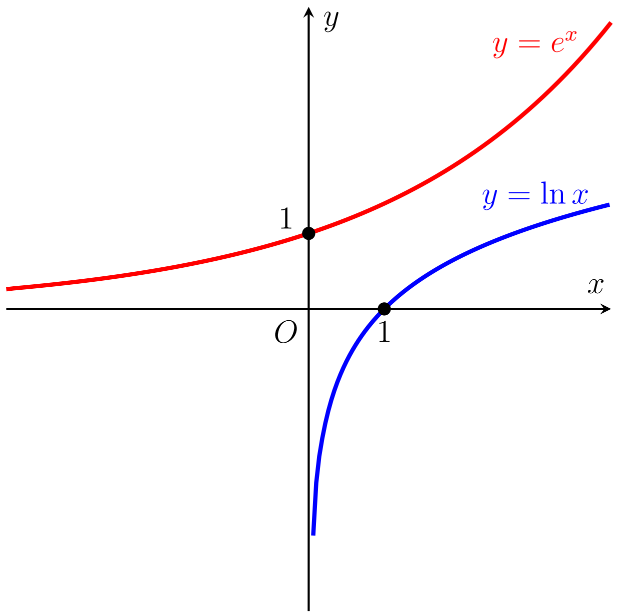
$$y = \log_a x$$

where $a > 0, \quad a \neq 1.$

$$a^{\log_a x} = \log_a(a^x) = x$$

We will use:

- $\log x$ for $\log_{10} x$ (common logarithm)
- $\ln x$ for $\log_e x$ (natural logarithm)



We can easily see that,

$$a^x \cdot a^y = a^{x+y} \quad \Rightarrow \quad \log_a(AB) = \log_a A + \log_a B$$

As a result of this,

- $\log_a \left(\frac{A}{B}\right) = \log_a A - \log_a B$
- $\log_a \left(\frac{1}{B}\right) = -\log_a B$
- $\log_a (A^r) = r \log_a A$

Any logarithm can be expressed in terms of the natural logarithm:

$$\log_a(x) = \frac{\ln x}{\ln a}$$

Any exponential can be expressed in terms of the natural exponential:

$$a^x = e^{x \ln a}$$

Example 3–6: Simplify $\log 360$.

Solution: First, we have to find factors of 360:

$$360 = 2^3 \cdot 3^2 \cdot 5$$

Now, we can use the properties of logarithms:

$$\begin{aligned} \log 360 &= \log 2^3 + \log 3^2 + \log 5 \\ &= 3\log 2 + 2\log 3 + 1 - \log 2 \\ &= 1 + 2\log 2 + 2\log 3 \end{aligned}$$

EXERCISES

Sketch the graphs of the following piecewise-defined functions:

3–1) $f(x) = \begin{cases} 2x & \text{if} \quad x < 5 \\ 10 & \text{if} \quad x \geqslant 5 \end{cases}$

3–2) $f(x) = \begin{cases} x+3 & \text{if} \quad x < 4 \\ x-1 & \text{if} \quad x \geqslant 4 \end{cases}$

Are the following functions polynomials?

3–3) $f(x) = 8x^4 + 1$

3–4) $f(x) = \dfrac{1-x}{x}$

3–5) $f(x) = \dfrac{1}{5}x + \dfrac{1}{3}$

3–6) $f(x) = 5x^5 - 3x^{2/3}$

Are the following functions one-to-one? Are they onto?

3–7) $f(x) = 2x$

3–8) $f(x) = x^3$

3–9) $f(x) = x^4 + x^2 + 1$

3–10) $f(x) = e^{2x}$

Find the inverse of the following functions.

3–11) $f(x) = 3x - 2$

3–12) $f(x) = \dfrac{x+2}{5x+4}$

3–13) $f(x) = \dfrac{1}{x}$

3–14) $f(x) = x^3 + 1$

Simplify the following:

3–15) $\log 400$

3–16) $\log 288$

3–17) $\log_9 27$

3–18) $\log_8 16$

3–19) $\log_2 1250$

3–20) $\log_3 \frac{\sqrt{3}}{81}$

3–21) $e^{2x+5\ln x}$

3–22) $\ln \frac{e}{\sqrt[3]{e}}$

3–23) $2^{3x+4\log_2 x}$

3–24) $3^{2\log_9 x}$

3–25) $5^{\log_{25} x}$

3–26) $10^{1+\log(2x)}$

Solve the following equations.

3–27) $5 = \left(5\sqrt{5}\right)^x$

3–28) $\log_x 12 = \frac{1}{2}$

3–29) $\log_x 77 = -1$

3–30) $\log_x 2 = 3$

3–31) $\log_x 64 = 4$

3–32) $\log_3 x = 5$

3–33) $\log_9(18x) = 2$

3–34) $\log_5 x = -\frac{1}{2}$

3–35) $\log(\log x) = 0$

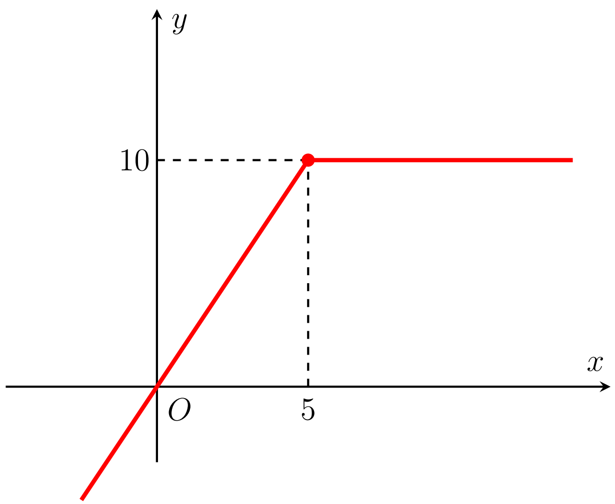
3–36) $\ln(\ln x) = 1$

3–37) $2^x = 100$

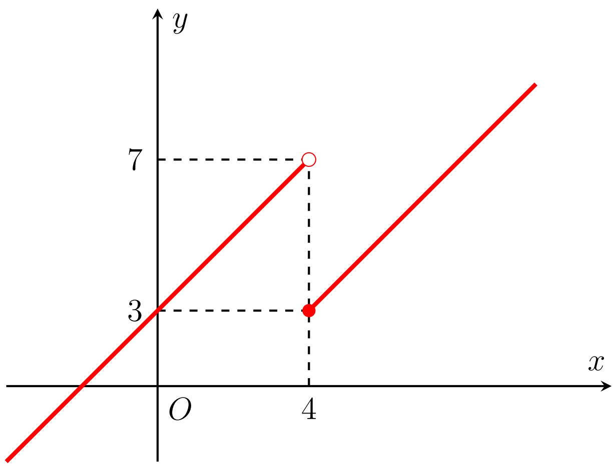
3–38) $2^{4x+4} = 8^{x-1}$

ANSWERS

3–1)



3–2)



3–3) Yes

3–4) No

3–5) Yes

3–6) No

3–7) One-to-one and onto.

3–8) One-to-one and onto.

3–9) Not one-to-one and not onto.

3–10) One-to-one and not onto.

3–11) $f^{-1}(x) = \dfrac{x+2}{3}$

3–12) $f^{-1}(x) = \dfrac{4x-2}{1-5x}$

3–13) $f^{-1}(x) = \dfrac{1}{x}$

3–14) $f^{-1}(x) = \sqrt[3]{x-1}$

3–15) $2 + 2\log 2$

3–16) $2\log 3 + 5\log 2$

3–17) $\frac{3}{2}$

3–18) $\frac{4}{3}$

3–19) $1 + 4\log_2 5$

3–20) $-\frac{7}{2}$

3–21) $x^5 e^{2x}$

3–22) $\frac{2}{3}$

3–23) $x^4 8^x$

3–24) x

3–25) $\sqrt{x}$

3–26) $20x$

3–27) $x = \frac{2}{3}$

3–28) $x = 144$

3–29) $x = \frac{1}{77}$

3–30) $x = 2^{1/3}$

3–31) $x = 2\sqrt{2}$

3–32) $x = 243$

3–33) $x = \frac{9}{2}$

3–34) $x = \frac{1}{\sqrt{5}}$

3–35) $x = 10$

3–36) $x = e^e$

3–37) $x = \frac{2}{\log 2}$

3–38) $x = -7$

Chapter 4

Limits

We say that $f(x)$ has the limit L at $x = a$ if $f(x)$ gets as close to L as we like, when x approaches a. (without getting equal to a)
We write this as:

$$\lim_{x \to a} f(x) = L$$

Example 4–1: Investigate the limit

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Solution:

x	f
0.9	1.9
0.99	1.99
0.999	1.999
$\vdots$	$\vdots$

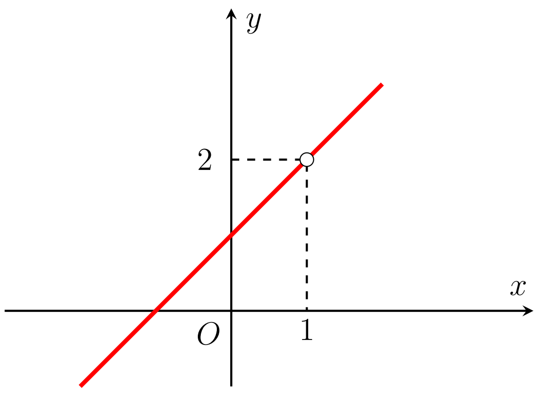
x	f
1.1	2.1
1.01	2.01
1.001	2.001
$\vdots$	$\vdots$

These results suggest that the limit is 2.

Actually, the function can be written as:

$$f(x) = \begin{cases} x + 1 & \text{if} \quad x \neq 1 \\ \text{undefined} & \text{if} \quad x = 1 \end{cases}$$

Its graph is:



Limit Laws: If both of the limits

$$\lim_{x\to a} f(x) = L \quad \text{and}$$

$$\lim_{x\to a} g(x) = M$$

exist, then:

- $\lim_{x\to a} f \pm g = L \pm M$
- $\lim_{x\to a} fg = LM$
- $\lim_{x\to a} \frac{f}{g} = \frac{L}{M}$ (if $M \neq 0$)
- $\lim_{x\to a} \sqrt[n]{f} = \sqrt[n]{L}$
- $\lim_{x\to a} f\big(g(x)\big) = f(M)$
  (If f is continuous at M)

Example 4–2: Evaluate the limit $\lim_{x\to 2} \frac{3}{x-2}$ (if it exists):

Solution: As x approaches 2, the function $\frac{3}{x-2}$ gets larger and larger without any bound. Therefore the limit does not exist. (Limit DNE.)

Example 4–3: Evaluate the following limit (if it exists):

$$\lim_{x\to -5} \frac{x^2+6x+5}{x^2+7x+10}$$

Solution:
$$\begin{aligned}\lim_{x\to -5} \frac{x^2+6x+5}{x^2+7x+10} &= \lim_{x\to -5} \frac{(x+5)(x+1)}{(x+5)(x+2)}\\ &= \lim_{x\to -5} \frac{(x+1)}{(x+2)}\\ &= \frac{4}{3}\end{aligned}$$

Example 4–4: Evaluate the following limit (if it exists):

$$\lim_{x\to 0} \frac{x^3-64}{x-4}$$

Solution: $\lim_{x\to 0} x^3 - 64 = -64$

$\lim_{x\to 0} x - 4 = -4$

Using limit laws, we obtain:

$$\lim_{x\to 0} \frac{x^3-64}{x-4} = \frac{-64}{-4} = 16$$

Example 4–5: Evaluate the limit $\lim_{x \to 4} \dfrac{x^3 - 64}{x - 4}$ if it exists.

Solution: This question is different.

Although the limit $\lim_{x \to 4} x - 4$ exists, it is zero, so we can NOT divide limit of the numerator by the limit of the denominator.

We have to use factorization:

$x^3 - 64 = (x - 4)(x^2 + 4x + 16)$

$$\begin{aligned}
\lim_{x \to 4} \frac{x^3 - 64}{x - 4} &= \lim_{x \to 4} \frac{(x - 4)(x^2 + 4x + 16)}{x - 4} \\
&= \lim_{x \to 4} (x^2 + 4x + 16) \\
&= 16 + 16 + 16 \\
&= 48
\end{aligned}$$

Example 4–6: Evaluate the limit (if it exists):

$$\lim_{x \to 5} \frac{1}{|x - 5|}$$

Solution: $\dfrac{1}{|x - 5|}$ increases without bounds as $x \to 5$.

Therefore limit does not exist.

Example 4–7: Evaluate the limit (if it exists):

$$\lim_{x \to -3} \frac{x^2 + 4x + 3}{x^2 + 5x + 6}$$

Solution:

$$\begin{aligned}
\lim_{x \to -3} \frac{x^2 + 4x + 3}{x^2 + 5x + 6} &= \lim_{x \to -3} \frac{(x + 3)(x + 1)}{(x + 3)(x + 2)} \\
&= \lim_{x \to -3} \frac{(x + 1)}{(x + 2)} \\
&= 2
\end{aligned}$$

Example 4–8: Evaluate the limit (if it exists)

$$\lim_{x \to 49} \frac{\sqrt{x} - 7}{x - 49}$$

Solution:

$$\begin{aligned}
\lim_{x \to 49} \frac{\sqrt{x} - 7}{x - 49} &= \lim_{x \to 49} \frac{\sqrt{x} - 7}{x - 49} \cdot \frac{\sqrt{x} + 7}{\sqrt{x} + 7} \\
&= \lim_{x \to 49} \frac{x - 49}{(x - 49)\left(\sqrt{x} + 7\right)} \\
&= \lim_{x \to 49} \frac{1}{\sqrt{x} + 7} \\
&= \frac{1}{14}
\end{aligned}$$

Example 4–9: Evaluate the limit (if it exists):

$$\lim_{x\to 2} \frac{x^3-7x+6}{x^2-5x+6}$$

Solution: This is of the form $\frac{0}{0}$, so, both the numerator and the denominator contain $(x-2)$.

Using polynomial division, we obtain:

$$\lim_{x\to 2} \frac{(x-2)(x^2+2x-3)}{(x-2)(x-3)}$$

For $x \neq 2$, this is:

$$= \lim_{x\to 2} \frac{(x^2+2x-3)}{(x-3)} = \frac{5}{-1} = -5$$

Example 4–10: Evaluate the limit (if it exists)

$$\lim_{x\to 8} \frac{2-\sqrt[3]{x}}{8-x}$$

Solution: Using the substitution $u = \sqrt[3]{x}$ we obtain:

$$\begin{aligned}
\lim_{x\to 8} \frac{2-\sqrt[3]{x}}{8-x} &= \lim_{u\to 2} \frac{2-u}{8-u^3} \\
&= \lim_{u\to 2} \frac{2-u}{(2-u)(4+2u+u^2)} \\
&= \lim_{u\to 2} \frac{1}{4+2u+u^2} \\
&= \frac{1}{12}
\end{aligned}$$

Example 4–11: Evaluate the limit (if it exists):

$$\lim_{x\to 3} \frac{x^2-9}{x^2-6x+9}$$

Solution: $= \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)^2} = \lim_{x\to 3} \frac{(x+3)}{(x-3)}$

Limit does not exist.

Example 4–12: Evaluate the following limit (if it exists):

$$\lim_{x\to 0} \frac{\sqrt{9+12x}-3}{x}$$

Solution: Multiply both numerator and denominator by the conjugate of the numerator:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{9+12x}-3}{x} &= \lim_{x\to 0} \frac{\sqrt{9+12x}-3}{x} \cdot \frac{\sqrt{9+12x}+3}{\sqrt{9+12x}+3} \\
&= \lim_{x\to 0} \frac{9+12x-9}{x\left(\sqrt{9+12x}+3\right)} \\
&= \lim_{x\to 0} \frac{12}{\sqrt{9+12x}+3} \\
&= 2
\end{aligned}$$

EXERCISES

Evaluate the following limits (if they exist):

4–1) $\lim_{x\to 2} \dfrac{x^3-8}{x-2}$

4–2) $\lim_{x\to -4} \dfrac{x^2+11x+28}{x^2+12x+32}$

4–3) $\lim_{x\to 5} \dfrac{x^2-25}{x^2-10x+25}$

4–4) $\lim_{x\to 1} \dfrac{x^3-x^2-x+1}{x^2-x}$

4–5) $\lim_{x\to 3} \dfrac{x^4-81}{x^3-27}$

4–6) $\lim_{x\to 0} \dfrac{\sqrt{x+16}-4}{x}$

4–7) $\lim_{x\to 3} \dfrac{x-3}{\sqrt{x+6}-3}$

4–8) $\lim_{x\to 64} \dfrac{\sqrt{x}-8}{x-64}$

4–9) $\lim_{x\to 0} \dfrac{\sqrt{2x+1}-3}{x}$

4–10) $\lim_{x\to 7} \dfrac{\sqrt{4x+8}-6}{x-7}$

Evaluate the following limits (if they exist):

4–11) $\lim_{x\to \infty} \dfrac{x(x^2-5x+14)}{7-4x^3}$

4–12) $\lim_{x\to \infty} \dfrac{3x^2+12x+9)}{(x^2-1)(x^2+1)}$

4–13) $\lim_{x\to \infty} \dfrac{2+3x-4x^4}{\sqrt{x}\big(1-17x+8x^3\big)}$

4–14) $\lim_{x\to \infty} \sqrt{x^2+6x}-x$

4–15) $\lim_{x\to \infty} \sqrt{x^2+4x}-\sqrt{x^2-10x+1}$

4–16) $\lim_{x\to \infty} \dfrac{x^4-16}{(2x-1)(2x+1)(x^2+1)}$

4–17) $\lim_{x\to \infty} \sqrt{x^2-12x+24}-\sqrt{x^2+10x+5}$

4–18) $\lim_{x\to \infty} \dfrac{1}{2x-\sqrt{4x^2-5x+6}}$

4–19) $\lim_{x\to 3} \dfrac{x^2-9}{x^2-7x+12}$

4–20) $\lim_{x\to 3} \dfrac{x^2-9}{\big(x-3\big)^2}$

Evaluate the following limits (if they exist):

4–21) $\lim\limits_{x\to 3} \dfrac{(x-3)^2}{x^2-9}$

4–22) $\lim\limits_{x\to 1} \dfrac{x^2-9}{x-3}$

4–23) $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

4–24) $\lim\limits_{x\to 1} \dfrac{1}{x^2-1}$

4–25) $\lim\limits_{x\to 1} \dfrac{x^3-1}{x^4-1}$

4–26) $\lim\limits_{x\to 4} \dfrac{2-\sqrt{x}}{4-x}$

4–27) $\lim\limits_{x\to 0} \dfrac{x^4-5x^2+12x+7}{5x^2+6}$

4–28) $\lim\limits_{x\to 2} \dfrac{x^2-7x+10}{x^2-5x+6}$

4–29) $\lim\limits_{x\to 4} \dfrac{x^2-7x+10}{x^2-5x+6}$

4–30) $\lim\limits_{x\to 6} \dfrac{x^2-5x+4}{x-6}$

Evaluate the following limits (if they exist):

4–31) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

4–32) $\lim\limits_{x\to -2} \dfrac{(x+2)^2}{x^4-16}$

4–33) $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-2}{x^2-9}$

4–34) $\lim\limits_{x\to 1} \dfrac{1-\sqrt{x}}{1-\sqrt[3]{x}}$

4–35) $\lim\limits_{x\to c} \dfrac{x^4-c^4}{x^3-c^3}$

4–36) $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{a+bx}-\sqrt{a-cx}}$

4–37) $\lim\limits_{x\to 1} \dfrac{x^n-1}{x-1}$

4–38) $\lim\limits_{x\to \infty} \dfrac{1}{\ln(x^2)}$

4–39) $\lim\limits_{x\to \infty} \dfrac{8e^x}{4+5e^x}$

4–40) $\lim\limits_{x\to \infty} \sqrt{2x^2-1}-\sqrt{x^2+1}$

ANSWERS

4–1) 12

4–2) $\frac{3}{4}$

4–3) Limit DNE.
(Limit does not exist.)

4–4) 0

4–5) 4

4–6) $\frac{1}{8}$

4–7) 6

4–8) $\frac{1}{16}$

4–9) Limit DNE.

4–10) $\frac{1}{3}$

4–11) $-\frac{1}{4}$

4–12) 0

4–13) $-\infty$

4–14) 3

4–15) 7

4–16) $\frac{1}{4}$

4–17) -11

4–18) $\frac{4}{5}$

4–19) -6

4–20) Limit DNE.

4–21) 0

4–22) 4

4–23) Limit DNE.

4–24) Limit DNE.

4–25) $\frac{3}{4}$

4–26) $\frac{1}{4}$

4–27) $\frac{7}{6}$

4–28) 3

4–29) -1

4–30) Limit DNE.

4–31) 1

4–32) 0

4–33) $\frac{1}{24}$

4–34) $\frac{3}{2}$

4–35) $\frac{4}{3}c$

4–36) $\frac{2\sqrt{a}}{b+c}$

4–37) n

4–38) 0

4–39) $\frac{8}{5}$

4–40) ∞

Chapter 5

One Sided Limits, Continuity

One Sided Limits: If x approaches a from right, taking values larger than a only, we denote this by $x \to a^+$. If $f(x)$ approaches L as $x \to a^+$, then we say that L is the right-hand limit of f at a.

$$\lim_{x \to a^+} f(x) = L$$

We define the left-hand limit of f at a similarly:

$$\lim_{x \to a^-} f(x) = L$$

Theorem: The limit $\lim_{x \to a} f(x) = L$ exists if and only if both one sided limits

$$\lim_{x \to a^+} f(x) \quad \text{and} \quad \lim_{x \to a^-} f(x)$$

exist and are equal to L.

Example 5–1: Consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if} \quad x < 1 \\ 5x - 2 & \text{if} \quad x > 1 \end{cases}$$

Find the limits $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1} f(x)$.

Solution: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x - 1 = 1$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 5x - 2 = 3$$

$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$ therefore $\lim_{x \to 1} f(x)$ does not exist.

Example 5–2: Find the limits $\lim_{x\to 3^+} f(x)$ and $\lim_{x\to 3^-} f(x)$ and graph the function:

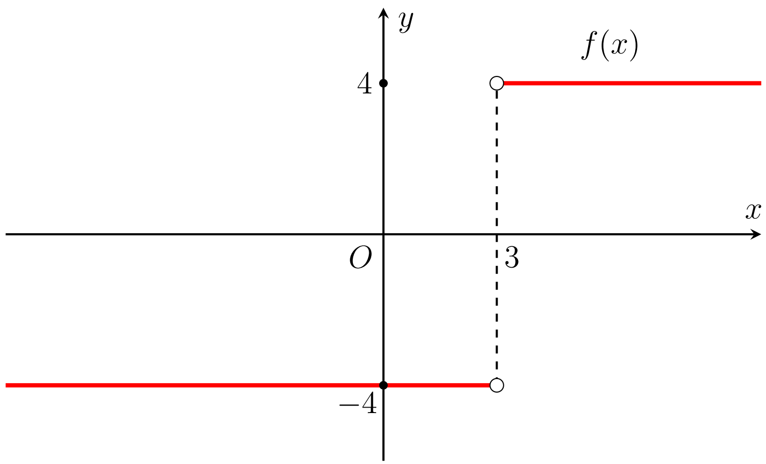
$$f(x) = \frac{4x-12}{|x-3|}$$

Solution: As $x \to 3^+$, $x - 3 > 0$ therefore $|x-3| = x-3$ and

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{4x-12}{x-3} = 4$$

Similarly,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{4x-12}{-(x-3)} = -4$$



We can see that left and right limits exist at $x = 3$.

But the limit $\lim_{x\to 3} f(x)$ does NOT exist.

Also, note that $f(3)$ is undefined.

Example 5–3: Let $f(x) = \begin{cases} 2 - x^2 & \text{if} \quad x < 0 \\ 7 & \text{if} \quad x = 0 \\ e^x + e^{-x} & \text{if} \quad x > 0 \end{cases}$

Find the limits $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0} f(x)$.

Solution: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2 - x^2 = 2$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^x + e^{-x} = 2$$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 2$ therefore $\lim_{x\to 0} f(x) = 2$.

(Note that the function value $f(0) = 7$ does not have any effect on the limit.)

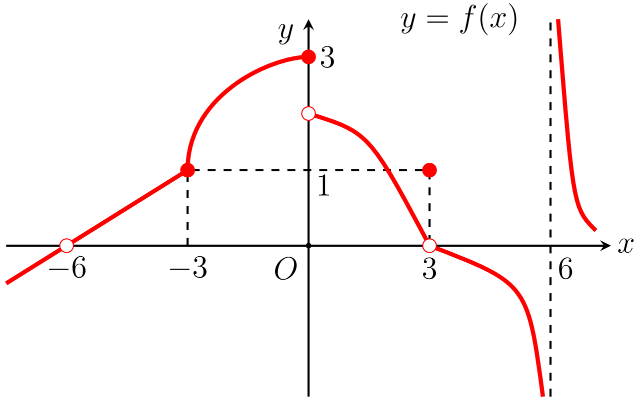
Example 5–4: Find the limit $\lim_{x\to 0^+} \ln x$ if it exists.

Solution: Checking the graph of $f(x) = \ln x$ we see that:

$$\lim_{x\to 0^+} \ln x = -\infty$$

Note that the question $\lim_{x\to 0} \ln x$ would be meaningless.

Example 5–5: Find the limits based on the function $f(x)$ in the figure: (If they exist.)



a) $\lim\limits_{x\to -6^-} f(x),\quad \lim\limits_{x\to -6^+} f(x),\quad \lim\limits_{x\to -6} f(x).$

b) $\lim\limits_{x\to -3^-} f(x),\quad \lim\limits_{x\to -3^+} f(x),\quad \lim\limits_{x\to -3} f(x).$

c) $\lim\limits_{x\to 0^-} f(x),\quad \lim\limits_{x\to 0^+} f(x),\quad \lim\limits_{x\to 0} f(x).$

d) $\lim\limits_{x\to 3^-} f(x),\quad \lim\limits_{x\to 3^+} f(x),\quad \lim\limits_{x\to 3} f(x).$

e) $\lim\limits_{x\to 6^-} f(x),\quad \lim\limits_{x\to 6^+} f(x),\quad \lim\limits_{x\to 6} f(x).$

Solution:

a) $0,\quad 0,\quad 0.$

b) $1,\quad 1,\quad 1.$

c) $3,\quad 2,\quad$ does not exist.

d) $0,\quad 0,\quad 0.$

e) $-\infty,\quad \infty,\quad$ does not exist.

Example 5–6: Evaluate the limit (if it exists)

$$\lim_{x\to 8^+} \frac{x^2-10x+16}{\sqrt{x-8}}$$

Solution: Note that square root of a negative number is not defined, so x should not take values less than 8.

Therefore the question

$$\lim_{x\to 8} \frac{x^2-10x+16}{\sqrt{x-8}}$$

would be meaningless.

Now if we factor $x^2-10x+16$ as:

$$\begin{aligned} x^2-10x+16 &= (x-8)(x-2) \\ &= \sqrt{x-8}\,\sqrt{x-8}\,(x-2) \end{aligned}$$

we obtain:

$$\begin{aligned} \lim_{x\to 8^+} \frac{x^2-10x+16}{\sqrt{x-8}} &= \lim_{x\to 8^+} \frac{\sqrt{x-8}\,\sqrt{x-8}\,(x-2)}{\sqrt{x-8}} \\ &= \lim_{x\to 8^+} \sqrt{x-8}\,(x-2) \\ &= 0 \end{aligned}$$

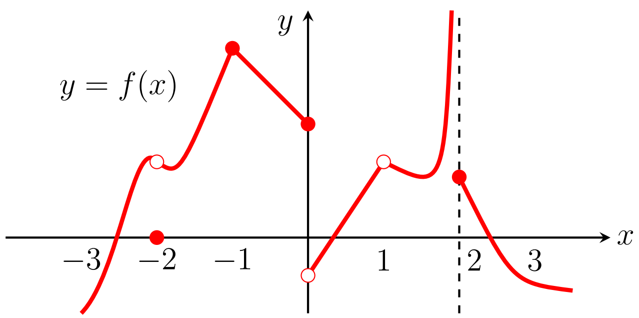
Continuity: We say that f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

In other words:

- f must be defined at a.
- $\lim_{x \to a} f(x)$ must exist.
- The limit must be equal to the function value.

Example 5–7: Determine the points where $f(x)$ is discontinuous:



Solution: $f(x)$ is discontinuous at:

- $x = -2$, limit and function value are different.
- $x = 0$, limit does not exist.
- $x = 1$, function is undefined.
- $x = 2$, limit does not exist.

Example 5–8: Let $f(x) = \begin{cases} 2x^2 + a & \text{if} \quad x < 2 \\ b & \text{if} \quad x = 2 \\ 3x - 2 & \text{if} \quad x > 2 \end{cases}$

Find a and b if $f(x)$ is continuous at $x = 2$.

Solution: $\lim_{x \to 2^-} f(x) = 8 + a$ and $\lim_{x \to 2^+} f(x) = 4$.

If f is continuous at $x = 2$, then

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$\Rightarrow \quad 8 + a = b = 4$$

We find $a = -4, \quad b = 4$.

Example 5–9: Let $f(x) = 2 + 12x - x^3 + 20x^4$. Find the points where $f(x)$ is discontinuous.

Solution: The given function is a polynomial. A polynomial function is continuous at all points.

Example 5–10: Let $f(x) = \dfrac{3x-2}{x^2+4}$. Find the points where $f(x)$ is discontinuous.

Solution: This is a rational function. A rational function is discontinuous only at the points where denominator is zero. But the equation

$$x^2 + 4 = 0$$

has no solutions. This means there is no discontinuity. In other words $f(x)$ is continuous on $\mathbb{R}$.

Example 5–11: Let $f(x) = \dfrac{x-2}{x^2-7x+10}$.

Find the points where $f(x)$ is discontinuous.

Solution: This is a rational function. So:

$$x^2 - 7x + 10 = 0 \quad \Rightarrow \quad x = 2,\ x = 5$$

$f(x)$ is discontinuous at $x = 2$ and $x = 5$.

Example 5–12: Let $f(x) = \begin{cases} \log\left(\dfrac{x}{2} + b\right) & \text{if} \quad x < 8 \\ x\left(\sqrt{x-8} + \dfrac{1}{4}\right) & \text{if} \quad x \geqslant 8 \end{cases}$

Find b if $f(x)$ is continuous at $x = 8$.

Solution: $\lim\limits_{x\to 8^+} f(x) = 2$

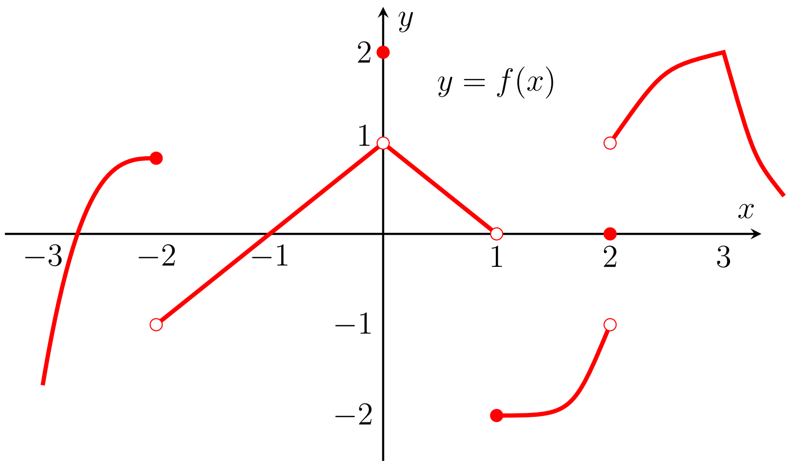
$\lim\limits_{x\to 8^-} f(x) = \log(4+b)$

If f is continuous, these limits must be equal.

$$\begin{aligned} \log(4+b) &= 2 \\ 4+b &= 100 \\ b &= 96 \end{aligned}$$

EXERCISES

5–1) Find the limits based on the figure:



a) $\lim\limits_{x\to -2^-} f(x)$, $\lim\limits_{x\to -2^+} f(x)$, $\lim\limits_{x\to -2} f(x)$.

b) $\lim\limits_{x\to -1^-} f(x)$, $\lim\limits_{x\to -1^+} f(x)$, $\lim\limits_{x\to -1} f(x)$.

c) $\lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 0} f(x)$.

d) $\lim\limits_{x\to 1^-} f(x)$, $\lim\limits_{x\to 1^+} f(x)$, $\lim\limits_{x\to 1} f(x)$.

e) $\lim\limits_{x\to 2^-} f(x)$, $\lim\limits_{x\to 2^+} f(x)$, $\lim\limits_{x\to 2} f(x)$.

f) $\lim\limits_{x\to 3^-} f(x)$, $\lim\limits_{x\to 3^+} f(x)$, $\lim\limits_{x\to 3} f(x)$.

5–2) Find the points where $f(x)$ of previous question is discontinuous.

Evaluate the following limits: (If they exist)

5–3) $\lim_{x\to 7^-} \frac{2}{x-7}$

5–4) $\lim_{x\to 7^+} \frac{2}{x-7}$

5–5) $\lim_{x\to 7^-} \frac{|x-7|}{x-7}$

5–6) $\lim_{x\to 7^+} \frac{|x-7|}{x-7}$

5–7) $\lim_{x\to 3^+} \sqrt{\frac{x-3}{x+3}}$

5–8) $\lim_{x\to 0^+} \frac{\sqrt{16+3x}-4}{x}$

5–9) $\lim_{x\to -2^+} \frac{|x^2-4|}{x+2}$

5–10) $\lim_{x\to -2^-} \frac{|x^2-4|}{x+2}$

5–11) $\lim_{x\to 0^+} \frac{2x^2+3x|x|}{x|x|}$

5–12) $\lim_{x\to 0^-} \frac{2x^2+3x|x|}{x|x|}$

Find all the discontinuities of the following functions:

5–13) $f(x) = \frac{x^2-2}{x^2-4}$

5–14) $f(x) = \frac{|x-a|}{x-a}$

5–15) $f(x) = \frac{x^2-5x+6}{x^2-4x+3}$

5–16) $f(x) = \frac{1}{e^{2x}-e^{3x}}$

5–17) $f(x) = \frac{x-5}{x^2-25}$

5–18) $f(x) = \frac{1}{1-|x|}$

5–19) $f(x) = \begin{cases} -1+x & \text{if } x \leqslant 0 \\ 1+x^2 & \text{if } x > 0 \end{cases}$

5–20) $f(x) = \begin{cases} 12x-20 & \text{if } x < 2 \\ 8 & \text{if } x = 2 \\ x^2 & \text{if } x > 2 \end{cases}$

Find the values of constants that will make the following functions continuous everywhere:

5–21) $f(x) = \begin{cases} a + bx^2 & \text{if} \quad x < 0 \\ b & \text{if} \quad x = 0 \\ 2 + e^{-x} & \text{if} \quad x > 0 \end{cases}$

5–22) $f(x) = \begin{cases} cx^2 - 2 & \text{if} \quad x \leqslant 2 \\ \dfrac{x}{c} & \text{if} \quad x > 2 \end{cases}$

5–23) $f(x) = \begin{cases} x^2 - c^2 & \text{if} \quad x \leqslant 1 \\ (x - c)^2 & \text{if} \quad x > 1 \end{cases}$

5–24) $f(x) = \begin{cases} e^{ax} & \text{if} \quad x \leqslant 0 \\ \ln\left(b + x^2\right) & \text{if} \quad x > 0 \end{cases}$

ANSWERS

5–1)

a) 1, -1, Does Not Exist.

b) 0, 0, 0.

c) 1, 1, 1.

d) 0, -2, DNE.

e) -1, 1, DNE.

f) 2, 2, 2.

5–2)

$x = -2$.

$x = 0$.

$x = 1$.

$x = 2$.

5–3) $-\infty$

5–4) ∞

5–5) -1

5–6) 1

5–7) 0

5–8) $\dfrac{3}{8}$

5–9) 4

5–10) -4

5–11) 5

5–12) 1

5–13) $x = 2$ and $x = -2$.

5–14) $x = a$.

5–15) $x = 1$, $x = 3$.

5–16) $x = 0$.

5–17) $x = -5$, $x = 5$.

5–18) $x = 1$ and $x = -1$.

5–19) $x = 0$.

5–20) $x = 2$.

5–21) $a = b = 3$

5–22) $c = 1$, or $c = -\dfrac{1}{2}$

5–23) $c = 0$, or $c = 1$

5–24) $b = e$, a is arbitrary.

Chapter 6

Derivatives

Definition and Notation: The derivative of the function $f(x)$ is the function $f'(x)$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\text{Or, equivalently:} \quad f'(x) = \lim_{a \to x} \frac{f(x) - f(a)}{x - a}$$

We can think of the derivative as

- The rate of change of a function f, or
- The slope of the curve of $y = f(x)$.

We will use $y', \quad f'(x), \quad \dfrac{dy}{dx}, \quad \dfrac{d}{dx}f(x)$ to denote derivatives and $f'(a), \quad \left.\dfrac{dy}{dx}\right|_{x=a}$ to denote their values at a certain point.

Note that derivative is a function, its value at a point is a number.

Higher Order Derivatives: We can find the derivative of the derivative of a function. It is called second derivative and denoted by:

$$y'', \quad f''(x), \quad \frac{d^2y}{dx^2}.$$

For third derivative, we use $f'''(x)$ but for fourth and higher derivatives, we use the notation $f^{(4)}(x), \ f^{(5)}(x)$ etc.

Example 6–1: Let $f(x) = 7x^3 - 18x$. Find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.

Solution:

$$\begin{aligned} f'(x) &= 21x^2 - 18 \\ f''(x) &= 42x \\ f'''(x) &= 42 \\ f^{(4)}(x) &= 0 \end{aligned}$$

Differentiation Formulas: Using the definition of derivative, we obtain:

- Derivative of a constant is zero, i.e.

$$\frac{dc}{dx} = 0$$

- Derivative of $f(x) = x$ is 1:

$$\frac{d}{dx} x = 1$$

- Derivative of $f(x) = x^2$ is $2x$:

$$\frac{d}{dx} x^2 = 2x$$

- Derivative of $f(x) = x^n$ is:

$$\frac{d}{dx} x^n = nx^{n-1}$$

- Derivative of $f(x) = \sqrt{x}$ is:

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

- If f is a function and c is a constant, then

$$(cf)' = cf'$$

- If f and g are functions, then

$$(f+g)' = f' + g'$$

Example 6–2: Evaluate the derivative of $f(x) = \dfrac{7x^3 - 18x}{x}$.

Solution: First we have to simplify:

$f(x) = 7x^2 - 18$

Then we use the differentiation rules:

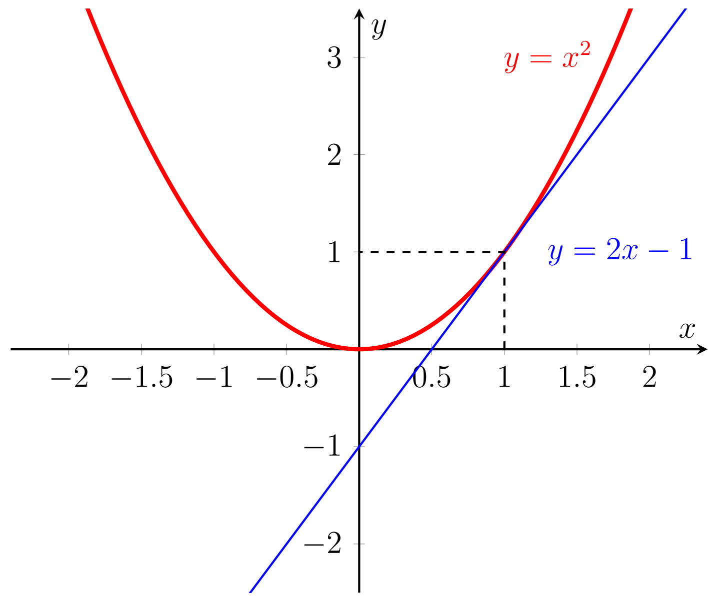
$f'(x) = 14x$

Example 6–3: Find the equation of the tangent line to the graph of $f(x) = x^2$ at the point $(1, 1)$.

Solution: $f'(x) = 2x \quad \Rightarrow \quad m = f'(1) = 2$

Using point slope equation $\big(y - y_0 = m(x - x_0)\big)$ we find the equation of the tangent line as:

$(y-1) = 2(x-1) \quad \Rightarrow \quad y = 2x - 1$



Differentiation Rules:

Product Rule: If f and g are differentiable at x, then fg is differentiable at x and

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

or more briefly:

$$(fg)' = f'g + fg'$$

Example 6–4: Find the derivative of $f(x) = (x^4 + 14x)(7x^3 + 17)$

Solution: $f'(x) = (4x^3 + 14)(7x^3 + 17) + (x^4 + 14x)\, 21x^2$

Reciprocal Rule: If f is differentiable at x and if $f(x) \neq 0$ then:

$$\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$$

Example 6–5: Using the reciprocal rule, find the derivative of $f(x) = \dfrac{1}{x^n}$.

Solution: $f'(x) = \dfrac{-nx^{n-1}}{x^{2n}} = -\dfrac{n}{x^{n+1}} = -n\, x^{-n-1}$

Example 6–6: Find the derivative of $f(x) = \dfrac{1}{8x^2 + 12x + 1}$.

Solution: $f'(x) = -\dfrac{16x + 12}{(8x^2 + 12x + 1)^2}$

Quotient Rule: If f and g are differentiable at x, and $g(x) \neq 0$ then $\dfrac{f}{g}$ is differentiable at x:

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

Example 6–7: Find the derivative of $f(x) = \dfrac{2x + 3}{5x^2 + 7}$.

Solution:
$$\begin{aligned} f'(x) &= \frac{2(5x^2 + 7) - 10x\,(2x + 3)}{(5x^2 + 7)^2} \\ &= \frac{-10x^2 - 30x + 14}{(5x^2 + 7)^2} \end{aligned}$$

Example 6–8: Find the derivative of $f(x) = \dfrac{1}{x^3 + x}$.

Solution: The quotient rule gives:

$$\begin{aligned} f'(x) &= \frac{0 \cdot (x^3 + x) - (3x^2 + 1) \cdot 1}{(x^3 + x)^2} \\ &= -\frac{3x^2 + 1}{(x^3 + x)^2} \end{aligned}$$

Alternatively, we can use reciprocal rule:

$$\begin{aligned} f'(x) &= \frac{-(x^3 + x)'}{(x^3 + x)^2} \\ &= -\frac{3x^2 + 1}{(x^3 + x)^2} \end{aligned}$$

Exponentials and Logarithms: The derivatives of exponential and logarithmic functions are

$$\frac{d}{dx}e^x = e^x \quad \text{and} \quad \frac{d}{dx}\ln x = \frac{1}{x}$$

e^x **is the only nonzero function whose derivative is itself.**

Example 6–9: Find the derivative of $f(x) = x^3 e^x$.

Solution: Using product rule,

$$f'(x) = 3x^2 e^x + x^3 e^x$$

Example 6–10: Find the derivative of $f(x) = e^x \ln x$.

Solution: Using product rule,

$$f'(x) = e^x \ln x + \frac{e^x}{x}$$

Example 6–11: Find the derivative of $f(x) = e^{-x}$.

Solution: We know that $e^{-x} = \dfrac{1}{e^x}$. Using reciprocal rule,

$$\begin{aligned} f'(x) &= \frac{-(e^x)'}{(e^x)^2} \\ &= -\frac{e^x}{e^{2x}} \\ &= -e^{-x} \end{aligned}$$

Example 6–12: Find the derivative of

$$f(x) = \frac{x^4}{e^x - x^2}$$

Solution: Using quotient rule,

$$\begin{aligned} f'(x) &= \frac{4x^3(e^x - x^2) - (e^x - 2x)x^4}{(e^x - x^2)^2} \\ &= \frac{4x^3e^x - 4x^5 - x^4e^x + 2x^5}{(e^x - x^2)^2} \\ &= \frac{(4x^3 - x^4)e^x - 2x^5}{(e^x - x^2)^2} \end{aligned}$$

Example 6–13: Find the derivative of

$$f(x) = \frac{1}{x - e^x + \ln x}$$

Solution: Using quotient rule,

$$\begin{aligned} f'(x) &= \frac{0 - \left(1 - e^x + \frac{1}{x}\right)}{(x - e^x + \ln x)^2} \\ &= -\frac{1 - e^x + \frac{1}{x}}{(x - e^x + \ln x)^2} \\ &= -\frac{x - xe^x + 1}{x(x - e^x + \ln x)^2} \end{aligned}$$

EXERCISES

Evaluate the derivatives of the following functions:

6–1) $f(x) = 1 - \sqrt{x}$

6–2) $f(x) = 4 + 3x - 12x^3$

6–3) $f(x) = x^{-1} + 4x^{-2}$

6–4) $f(x) = \dfrac{1}{\sqrt[4]{x}}$

6–5) $f(x) = \dfrac{1}{x} - \dfrac{2}{x^2}$

6–6) $f(x) = 20x^{-4} + 4x^{1/4}$

6–7) $f(x) = \dfrac{x^3 - x}{\sqrt{x}}$

6–8) $f(x) = \dfrac{2x^4 - x^3 - 1}{x}$

6–9) $f(x) = (x^2 + 2)(x^2 - 3)$

6–10) $f(x) = \dfrac{x}{x^2 + 4}$

6–11) $f(x) = \dfrac{x^2 + 12}{5x - 2}$

6–12) $f(x) = \dfrac{x^{3/2} + x^{-1/2}}{\sqrt{x} + \dfrac{1}{\sqrt{x}}}$

Evaluate the derivatives of the following functions:

6–13) $f(x) = x^{12}e^x$

6–14) $f(x) = x^2 \ln\left(x^3\right)$

6–15) $f(x) = \dfrac{5x}{\ln x}$

6–16) $f(x) = \dfrac{e^x}{1 + x^2}$

6–17) $f(x) = \dfrac{1}{1 + 2x + 3e^x}$

6–18) $f(x) = \dfrac{1}{e^x + 2\ln x}$

6–19) $f(x) = x^4 e^x \ln x$

6–20) $f(x) = \left(x + e^x\right)\left(x^2 + \ln x\right)$

6–21) $f(x) = \dfrac{4x^2 - 5x}{2e^x - 3x}$

6–22) $f(x) = \dfrac{1}{\ln\left(4x\right)}$

6–23) $f(x) = e^x e^x e^x$

6–24) $f(x) = \dfrac{2 - 3\ln x}{5\ln x + 1}$

ANSWERS

6–1) $f'(x) = \dfrac{-1}{2\sqrt{x}}$

6–2) $f'(x) = 3 - 36x^2$

6–3) $f'(x) = -x^{-2} - 8x^{-3}$

6–4) $f'(x) = -\dfrac{1}{4}x^{-5/4}$

6–5) $f'(x) = -\dfrac{1}{x^2} + \dfrac{4}{x^3}$

6–6) $f'(x) = -80x^{-5} + x^{-3/4}$

6–7) $f'(x) = \dfrac{5}{2}x^{3/2} - \dfrac{1}{2}x^{-1/2}$

6–8) $f'(x) = 6x^2 - 2x + \dfrac{1}{x^2}$

6–9) $f'(x) = 4x^3 - 2x$

6–10) $f'(x) = \dfrac{4 - x^2}{(x^2+4)^2}$

6–11) $f'(x) = \dfrac{5x^2 - 4x - 60}{(5x-2)^2}$

6–12) $f'(x) = \dfrac{x^2 + 2x - 1}{(x+1)^2}$

6–13) $f'(x) = 12x^{11}e^x + x^12e^x$

6–14) $f'(x) = 6x\ln x + 3x$

6–15) $f'(x) = \dfrac{5\ln x - 5}{\ln^2 x}$

6–16) $f'(x) = \dfrac{e^x(1 + x^2 - 2x)}{(1+x^2)^2}$

6–17) $f'(x) = -\dfrac{2 + 3e^x}{(1 + 2x + 3e^x)^2}$

6–18) $f'(x) = -\dfrac{e^x + \frac{2}{x}}{(e^x + 2\ln x)^2}$

6–19) $f'(x) = x^3e^x\big(4\ln x + x\ln x + 1\big)$

6–20) $f'(x) = \big(1 + e^x\big)\big(x^2 + \ln x\big) + \big(x + e^x\big)\left(2x + \dfrac{1}{x}\right)$

6–21) $f'(x) = \dfrac{(8x-5)(2e^x - 3x) - (2e^x - 3)(4x^2 - 5x)}{(2e^x - 3x)^2}$

6–22) $f'(x) = -\dfrac{1}{x\ln^2(4x)}$

6–23) $f'(x) = 3e^{3x}$

6–24) $f'(x) = -\dfrac{13}{\big(5\ln x + 1\big)^2}$

Chapter 7

Chain Rule

Chain Rule: If f and g are differentiable then $f(g(x))$ is also differentiable and

$$\Big[f\big(g(x)\big)\Big]' = f'\big(g(x)\big) \cdot g'(x)$$

or more briefly :

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Example 7–1: Find $\dfrac{d}{dx}\big(3x^2+1\big)^5$.

Solution: Here $u = 3x^2 + 1$ and $y = u^5$. Using the above formula, we obtain:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 5u^4 \cdot 6x \\ &= 5(3x^2+1)^4 \cdot 6x \\ &= 30x(3x^2+1)^4 \end{aligned}$$

Example 7–2: Find $f'(x)$ where $f(x) = e^{x^5}$.

Solution: Here $u = x^5$ and $f = e^u$. Using the chain rule formula, we obtain:

$$f'(x) = e^{x^5} \cdot 5x^4$$

Example 7–3: Find $f'(x)$ where $f(x) = \ln(1 + 2x + 5x^2)$.

Solution: Here $u = 1 + 2x + 5x^2$ and $f = \ln u$. Using chain rule:

$$\begin{aligned} f'(x) = \frac{df}{dx} &= \frac{df}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{u} \cdot (2 + 10x) \\ &= \frac{1}{1+2x+5x^2} \cdot (2+10x) \\ &= \frac{2+10x}{1+2x+5x^2} \end{aligned}$$

Example 7–4: Find the derivatives of the following functions using chain rule:

a) $f(x) = \sqrt{2x-3}$

b) $f(x) = \left(x^3 + e^x\right)^7$

c) $f(x) = \ln\left(\dfrac{x+1}{2x+1}\right)$

Solution:

a) Choose $u = 2x - 3 \quad \Rightarrow \quad \dfrac{du}{dx} = 2$

$$\begin{aligned} f'(x) &= \frac{1}{2}(2x-3)^{-1/2} \cdot 2 \\ &= \frac{1}{\sqrt{2x-3}} \end{aligned}$$

b) Choose $u = x^3 + e^x \quad \Rightarrow \quad \dfrac{du}{dx} = 3x^2 + e^x$

$$f'(x) = 7\left(x^3 + e^x\right)^6 \cdot (3x^2 + e^x)$$

c) Choose $u = \dfrac{x+1}{2x+1}$

$$\Rightarrow \quad \frac{du}{dx} = \frac{2x+1-2(x+1)}{(2x+1)^2} = -\frac{1}{(2x+1)^2}$$

$$\begin{aligned} f'(x) &= \frac{1}{\left(\dfrac{x+1}{2x+1}\right)} \cdot \frac{-1}{(2x+1)^2} \\ &= -\frac{1}{(x+1)(2x+1)} \end{aligned}$$

Example 7–5: Find the derivatives of the following functions using chain rule:

a) $f(x) = e^{ax}$

b) $f(x) = \ln\left(ax\right)$

c) $f(x) = e^{x^2 - x}$

d) $f(x) = \ln\left(x^8\right)$

Solution:

a) $u = ax \quad \Rightarrow \quad \dfrac{du}{dx} = a$

$f'(x) = ae^{ax}$

b) $u = ax \quad \Rightarrow \quad \dfrac{du}{dx} = a$

$f'(x) = \dfrac{1}{ax} \cdot a = \dfrac{1}{x}$

c) $u = x^2 - x \quad \Rightarrow \quad \dfrac{du}{dx} = 2x - 1$

$f'(x) = (2x-1)e^{x^2 - x}$

d) $u = x^8 \quad \Rightarrow \quad \dfrac{du}{dx} = 8x^7$

$f'(x) = \dfrac{1}{x^8} \cdot 8x^7 = \dfrac{8}{x}$

Logarithmic Differentiation: Logarithm transforms products into sums. This helps in finding derivatives of some complicated functions.

For example if

$$y = \frac{(x^3+1)(x^2-1)}{x^8+6x^4+1}$$

then

$$\ln y = \ln(x^3+1) + \ln(x^2-1) - \ln(x^8+6x^4+1)$$

Derivative of both sides gives:

$$\frac{y'}{y} = \frac{3x^2}{x^3+1} + \frac{2x}{x^2-1} - \frac{8x^7+24x^3}{x^8+6x^4+1}$$

Example 7–6: Find the derivative of the function

$$y = f(x) = x^x$$

Solution: We can not use the power rule or product rule here. We have to use logarithms.

$$\begin{aligned} \ln y &= x \ln x \\ (\ln y)' &= \ln x + x \cdot \frac{1}{x} \\ \frac{y'}{y} &= \ln x + 1 \\ y' &= (\ln x + 1)\, x^x \end{aligned}$$

Example 7–7: Find the derivative of the function:

$$y = x^{\ln x}$$

Solution:

$$\begin{aligned} \ln y &= \ln x \cdot \ln x = \ln^2 x \\ (\ln y)' &= 2 \ln x \cdot \frac{1}{x} \\ \frac{y'}{y} &= \frac{2 \ln x}{x} \\ y' &= \frac{2 \ln x}{x} \cdot x^{\ln x} = 2x^{\ln x - 1} \ln x \end{aligned}$$

Example 7–8: Find the derivative of the function:

$$y = (x + e^x)^{\ln x}$$

Solution:

$$\begin{aligned} \ln y &= \ln x \ln(x + e^x) \\ (\ln y)' &= \frac{1}{x} \ln(x+e^x) + \frac{1+e^x}{x+e^x} \ln x \\ \frac{y'}{y} &= \frac{\ln(x+e^x)}{x} + \frac{1+e^x}{x+e^x} \ln x \end{aligned}$$

$$y' = \left(\frac{\ln(x+e^x)}{x} + \frac{1+e^x}{x+e^x} \ln x \right) (x+e^x)^{\ln x}$$

EXERCISES

Evaluate the derivatives of the following functions using chain rule:

7–1) $f(x) = \left(1 + x^4\right)^2$

7–2) $f(x) = e^{x^3}$

7–3) $f(x) = \ln\left(1 + x^2\right)$

7–4) $f(x) = \left(5 + x + 2x^3\right)^7$

7–5) $f(x) = \dfrac{x}{\sqrt{3x^2 + 2}}$

7–6) $f(x) = \dfrac{1}{(x^2 - 4x)^3}$

7–7) $f(x) = \left(\dfrac{2x}{x - 1}\right)^5$

7–8) $f(x) = \left(e^{3x} + 1\right)^5$

7–9) $f(x) = \sqrt{1 + \ln x}$

7–10) $f(x) = \sqrt{x^2 + 2e^{3x}}$

7–11) $f(x) = 4^{x^2+5x}$

7–12) $f(x) = xe^x \log_3\left(x + x^4\right)$

Find f'':

7–13) $f(x) = 5^{2x}$

7–14) $f(x) = \ln\left(3x\right)$

7–15) $f(x) = \sqrt{2 + x}$

7–16) $f(x) = x^7 e^{-x}$

Find f' using logarithmic differentiation:

7–17) $f(x) = \left(1 + 2x\right)^7 \left(x^3 + 1\right)^4$

7–18) $f(x) = \dfrac{\left(3x^4 + x^2\right)^6}{\left(1 + x + x^2\right)^8}$

7–19) $f(x) = (\ln x)^x$

Find the equation of the line tangent to $f(x)$ at x_0:

7–20) $f(x) = 2x^2 - 8x + 4, \quad x_0 = 2.$

7–21) $f(x) = x\sqrt{2x + 4}, \quad x = 0.$

7–22) $f(x) = x^2(1 - x)^2, \quad x = 2.$

7–23) $f(x) = \dfrac{1}{1 + x^2}, \quad x = 0.$

Evaluate the derivatives of the following functions at the point $x = a$. In other words, find the value of $f'(a)$.

7–24) $f(x) = \dfrac{2x^2 - 3x + 12}{x}, \quad a = 2.$

7–25) $f(x) = x^{3/5}, \quad a = 32.$

7–26) $f(x) = \dfrac{5 + 3x^2}{8 + 4x}, \quad a = 0.$

7–27) $f(x) = \dfrac{\ln x}{x^4}, \quad a = 1.$

7–28) $f(x) = \big(1 + 2x\big)e^x, \quad a = 0.$

7–29) $f(x) = \sqrt{10 - e^{-x}}, \quad a = 0.$

7–30) $f(x) = \ln\left(\dfrac{x - 2}{3x - 3}\right), \quad a = 5.$

7–31) $f(x) = \left(2x + \dfrac{3}{x}\right)^2, \quad a = \dfrac{1}{2}.$

7–32) $f(x) = \big(4x + e^{5x}\big)^3, \quad a = 0.$

7–33) $f(x) = \dfrac{1}{2 + 4x + 8e^{2x}}, \quad a = 0.$

7–34) $f(x) = x \ln \sqrt{1 + 2x}, \quad a = 1.$

7–35) $f(x) = \ln\left(\dfrac{xe^x}{1 + x^2}\right), \quad a = 2.$

ANSWERS

7–1) $f'(x) = 8\big(1 + x^4\big)x^3$

7–2) $f'(x) = 3x^2 e^{x^3}$

7–3) $f'(x) = \dfrac{2x}{1 + x^2}$

7–4) $f'(x) = 7\big(5 + x + 2x^3\big)^6\big(1 + 6x^2\big)$

7–5) $f'(x) = \dfrac{2}{(3x^2 + 2)^{3/2}}$

7–6) $f'(x) = \dfrac{12 - 6x}{(x^2 - 4x)^4}$

7–7) $f'(x) = 5\left(\dfrac{2x}{x - 1}\right)^4 \dfrac{-2}{(x - 1)^2} = -\dfrac{160x^4}{(x - 1)^6}$

7–8) $f'(x) = 15(e^{3x} + 1)^4 e^{3x}$

7–9) $f'(x) = \dfrac{1}{2x\sqrt{1 + \ln x}}$

7–10) $f'(x) = \dfrac{2x + 6e^{3x}}{2\sqrt{x^2 + 2e^{3x}}}$

7–11) $f'(x) = 4^{x^2 + 5x}\,\big(2x + 5\big)$

7–12) $f'(x) = (e^x + xe^x)\log_3\big(x + x^4\big) + xe^x \dfrac{1 + 4x^3}{(x + x^4)\ln 3}$

7–13) $f''(x) = (4\ln^2 5)\, 5^{2x}$

7–14) $f''(x) = -\dfrac{1}{x^2}$

7–15) $f''(x) = \dfrac{1}{4\,(2+x)^{3/2}}$

7–16) $f''(x) = \left(42x^5 - 14x^6 + x^7\right) e^{-x}$

7–17) $f'(x) = \left(1+2x\right)^7 \left(x^3+1\right)^4 \left[\dfrac{14}{1+2x} + \dfrac{12x^2}{x^3+1}\right]$

7–18) $f'(x) = \dfrac{\left(3x^4+x^2\right)^6}{\left(1+x+x^2\right)^8}\left[\dfrac{6(12x^3+2x)}{3x^4+x^2} - \dfrac{8(1+2x)}{1+x+x^2}\right]$

7–19) $f'(x) = (\ln x)^x \left[\ln\left(\ln x\right) + \dfrac{1}{\ln x}\right]$

7–20) $y = -4$

7–21) $y = 2x$

7–22) $y = -4x + 12$

7–23) $y = 1$

7–24) $f'(2) = -1$

7–25) $f'(32) = \dfrac{3}{20}$

7–26) $f'(0) = -\dfrac{5}{16}$

7–27) $f'(1) = 1$

7–28) $f'(0) = 3$

7–29) $f'(0) = \dfrac{1}{6}$

7–30) $f'(5) = \dfrac{1}{12}$

7–31) $f'\left(\dfrac{1}{2}\right) = -140$

7–32) $f'(0) = 27$

7–33) $f'(0) = -\dfrac{1}{5}$

7–34) $f'(1) = \dfrac{1}{2}\ln 3 + \dfrac{1}{3}$

7–35) $f'(-1) = \dfrac{7}{10}$

Chapter 8

Implicit Differentiation

An equation involving x and y may define y as a function of x. This is called an implicit function. For example, the following equations define y implicitly.

- $x^2 + y^2 = 1$,
- $ye^y + 2x - \ln y = 0$,
- $3xy + x^2y^3 + x = 5$,
- $e^x + e^y = \sqrt{x + 2y}$,

The following equations define y explicitly.

- $y = x^3 - 5x^2$,
- $y = \ln\left(x^2 - e^x\right)$,
- $y = x^3 + \sqrt{x} + xe^x$,
- $y = \dfrac{1}{1 + e^{x^2 - x}}$,

The derivative of y can be found without solving for y. This is called implicit differentiation. The main idea is:

- Differentiate with respect to x.
- Solve for y'.

Example 8–1: Find y' using the equation $y + y^3 = 3x^2 + 1$.

Solution: Find the derivative with respect to x:

$$y' + 3y^2y' = 6x$$

Therefore $y' = \dfrac{6x}{1 + 3y^2}$

Remark: Note that we are using chain rule here. For example, derivative of y^n is:

$$\begin{aligned} \frac{d(y^n)}{dx} &= \frac{d(y^n)}{dy}\,\frac{dy}{dx} \\ &= ny^{n-1}\,y' \end{aligned}$$

Example 8–2: Find the slope of the tangent line to the curve

$$x^2 + y^2 = 4 \quad \text{at the point} \quad \left(1,\ \sqrt{3}\right)$$

Solution: Let's differentiate both sides with respect to x:

$$\begin{aligned} x^2 + y^2 &= 4 \\ \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}\left(4\right) \\ 2x + 2y\,\frac{dy}{dx} &= 0 \\ 2yy' &= -2x \\ y' &= -\frac{x}{y} \end{aligned}$$

Therefore at the point $\left(1,\ \sqrt{3}\right)$:

$$y' = -\frac{1}{\sqrt{3}}$$

An alternative method is to express y in terms of x explicitly as

$$y = \sqrt{4 - x^2}$$

and then differentiate as:

$$y' = \frac{1}{2}(4 - x^2)^{-1/2} \cdot (-2x)$$

and then insert $x = 1$, but usually this is not possible.

Example 8–3: Find y' using implicit differentiation where

$$xy + x^3y^2 = 5y$$

Solution: Let's differentiate both sides with respect to x. Note that we are also using product rule:

$$\begin{aligned} y + xy' + 3x^2y^2 + x^3 2yy' &= 5y' \\ xy' + 2x^3yy' - 5y' &= -y - 3x^2y^2 \\ \left(x + 2x^3y - 5\right)y' &= -y - 3x^2y^2 \\ y' &= -\frac{y + 3x^2y^2}{x + 2x^3y - 5} \end{aligned}$$

Example 8–4: Find y' using implicit differentiation where

$$x^2e^y + y = e^{3x}$$

Solution: Let's differentiate both sides with respect to x:

$$\begin{aligned} 2xe^y + x^2e^yy' + y' &= 3e^{3x} \\ \left(x^2e^y + 1\right)y' &= 3e^{3x} - 2xe^y \\ y' &= \frac{3e^{3x} - 2xe^y}{x^2e^y + 1} \end{aligned}$$

Example 8–5: Find y' using implicit differentiation where

$$\ln(x+3y) = \frac{1}{x^2}$$

Solution:
$$\begin{aligned}\frac{1}{x+3y}\cdot(1+3y') &= \frac{-2}{x^3}\\ 1+3y' &= \frac{-2(x+3y)}{x^3}\\ 3y' &= -\frac{2}{x^2}-\frac{6y}{x^3}-1\\ y' &= -\frac{2}{3x^2}-\frac{2y}{x^3}-\frac{1}{3}\end{aligned}$$

Example 8–6: Find y' using implicit differentiation where

$$ye^{xy}+x^4\ln x = e^{3x}$$

Solution: First we use product rule, then chain rule:

$$y'e^{xy}+y\big(y+xy'\big)e^{xy}+4x^3\ln x+x^3 = 3e^{3x}$$

$$e^{xy}y'+xye^{xy}y' = 3e^{3x}-4x^3\ln x-x^3-y^2e^{xy}$$

$$y' = \frac{3e^{3x}-4x^3\ln x-x^3-y^2e^{xy}}{e^{xy}+xye^{xy}}$$

$$y' = \frac{3e^{3x}-4x^3\ln x-x^3-y^2e^{xy}}{(1+xy)\,e^{xy}}$$

Example 8–7: Find the slope of the tangent line to the curve

$$x^8+4x^2y^2+y^8 = 6 \quad \text{at the point} \quad (1,1)$$

Solution: Using implicit differentiation we obtain:

$$8x^7+8xy^2+8x^2yy'+8y^7y' = 0$$

$$x^7+xy^2+\Big(x^2y+y^7\Big)y' = 0$$

$$\Rightarrow \quad y' = \frac{-x^7-xy^2}{x^2y+y^7}$$

At $(1,1)$ the slope is: $y' = \dfrac{-2}{2} = -1.$

Example 8–8: Find y' at $(0,0)$ where

$$(1+x+2y)e^y+3xe^x = 1+x^2+y^2$$

Solution: Using implicit differentiation we obtain:

$$(1+2y')e^y+(1+x+2y)e^yy'+3e^x+3xe^x = 2x+2yy'$$

$$\big(2e^y+(1+x+2y)e^y-2y\big)y' = 2x-e^y-3e^x-3xe^x$$

$$y' = \frac{2x-e^y-3(1+x)e^x}{(3+x+2y)e^y-2y}$$

$$y'(0,0) = -\frac{4}{3}.$$

EXERCISES

Find y' using implicit differentiation:

8–1) $x^2y^3 + 3xy^2 + y = 5$

8–2) $xye^x + (x + 2y)^2 = x$

8–3) $(x^2 + y)^2 = y^3$

8–4) $x = y + y^{2/3}$

8–5) $(1 + e^{-x})^2 = \ln(x + y)$

8–6) $\ln y = y^3 + \ln x$

8–7) $e^{xy} = x + 2y$

8–8) $x^2 + \ln y = 3xy$

Find y' using implicit differentiation:

8–9) $x^2y = e^y$

8–10) $x^4 + y^4 = 3x + 5y$

8–11) $xy^2 = 1 + \ln(xy)$

8–12) $e^y + x^2e^x = 18$

8–13) $y^2 \ln y = x^3e^x$

8–14) $\sqrt{5x + y^3} + xy = 12$

8–15) $\dfrac{2}{x} + \dfrac{7}{y} = 9$

8–16) $x^{1/3} + y^{1/5} = y$

Find y' at the indicated point using implicit differentiation:

8–17) $(1+2x+3y)^2 = 13x\ln x + 7y^5 + 29$ at $(1,1)$

8–18) $3x - 2y + 8x^2 + 5y^2 + 9e^{9x} + 7e^{2y} = 16$ at $(0,0)$

8–19) $x^2y^2 + 2xy^3 + y - 10x + 11 = 0$ at $(2,1)$

8–20) $xy^4 + 3y^5 + x - 3y^3 = 0$ at $(0,1)$

8–21) $2x - 4y^4 + x^2y^6 + 11y^3 = 0$ at $(3,-1)$

8–22) $\sqrt{11+y^2} - 12xy + 2y^2 + 4x = 0$ at $(1,5)$

8–23) $xe^x - ye^y + xy - 1 = 0$ at $(1,1)$

8–24) $\ln(xy) + xy^2 - \ln 3x - 6y = 0$ at $(2,3)$

ANSWERS

8–1) $y' = -\dfrac{2xy^3+3y^2}{3x^2y^2+6xy+1}$

8–2) $y' = -\dfrac{ye^x + xye^x + 2x + 4y - 1}{xe^x + 4x + 8y}$

8–3) $y' = \dfrac{4x^3+4xy}{3y^2-2x^2-2y}$

8–4) $y' = \dfrac{1}{1+\frac{2}{3}y^{-1/3}}$

8–5) $y' = -2(x+y)e^{-x}(1+e^{-x}) - 1$

8–6) $y' = \dfrac{y}{x(1-3y^3)}$

8–7) $y' = \dfrac{1-ye^{xy}}{xe^{xy}-2}$

8–8) $y' = \dfrac{3y^2-2xy}{1-3xy}$

8–9) $y' = \dfrac{2xy}{e^y - x^2}$

8–10) $y' = \dfrac{4x^3 - 3}{5 - 4y^4}$

8–11) $y' = \dfrac{y - xy^3}{2x^2y^2 - x}$

8–12) $y' = -(2x + x^2)e^{x-y}$

8–13) $y' = \dfrac{3x^2e^x + x^3e^x}{2y\ln y + y}$

8–14) $y' = -\dfrac{2y\sqrt{5x + y^3} + 5}{2x\sqrt{5x + y^3} + 3y^2}$

8–15) $y' = -\dfrac{2y^2}{7x^2}$

8–16) $y' = \dfrac{5y^{4/5}}{3x^{2/3}\left(5y^{4/5} - 1\right)}$

8–17) $y'\Big|_{(1,1)} = -11$

8–18) $y'\Big|_{(0,0)} = -7$

8–19) $y'\Big|_{(2,1)} = \dfrac{4}{21}$

8–20) $y'\Big|_{(0,1)} = -\dfrac{1}{3}$

8–21) $y'\Big|_{(3,-1)} = \dfrac{8}{5}$

8–22) $y'\Big|_{(1,5)} = \dfrac{336}{53}$

8–23) $y'\Big|_{(1,1)} = \dfrac{2e + 1}{2e - 1}$

8–24) $y'\Big|_{(2,3)} = -\dfrac{27}{19}$

Chapter 9

L'Hôpital's Rule

Some limits like $\frac{0}{0}$, $\frac{\infty}{\infty}$, ... etc. are called indeterminate forms. These limits may turn out to be definite numbers, or infinity, or may not exist. Note that when we say $\frac{0}{0}$ we do not mean dividing the number 0 by the number 0. This would be undefined.

$\frac{0}{0}$ is a notation we use to denote the limit

$$\lim_{x \to a} \frac{f}{g}$$

where $\lim_{x \to a} f = 0$, $\lim_{x \to a} g = 0$. The case $\frac{\infty}{\infty}$ is similar to this.

For example, consider the limits

$$\lim_{x \to 0} \frac{x^5}{x^2}, \qquad \lim_{x \to 0} \frac{x^5}{x^7}, \qquad \lim_{x \to 0} \frac{3x^5}{4x^5}$$

All are of the form $\frac{0}{0}$ but their results are 0, ∞ and $\frac{3}{4}$.

Consider the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Assume that this is an indeterminate form of the type:

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

Suppose $g'(x) \neq 0$ on an open interval containing a (except possibly at $x = a$).

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if this limit exists, or is $\pm\infty$.

This is called the **L'Hôpital's Rule**.

Example 9–1: Evaluate the following limit: (if it exists.)

$$\lim_{x\to 0} \frac{e^x - 1}{2x}$$

Solution: This limit is in the form $\frac{0}{0}$, so we will use L'Hôpital's rule:

$$\lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2}$$

To evaluate this limit, insert $x = 0$ to obtain:

$$\lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

Example 9–2: Evaluate the following limit: (if it exists.)

$$\lim_{x\to 0} \frac{e^{3x} - e^x - 2x}{x^2}$$

Solution: Limit is in the form $\frac{0}{0}$ $\Rightarrow$ use L'Hôpital's rule:

$$\lim_{x\to 0} \frac{e^{3x} - e^x - 2x}{x^2} = \lim_{x\to 0} \frac{3e^{3x} - e^x - 2}{2x}$$

This second limit is also in the form $\frac{0}{0}$, so we will use L'Hôpital's rule once more:

$$\lim_{x\to 0} \frac{3e^{3x} - e^x - 2}{2x} = \lim_{x\to 0} \frac{9e^{3x} - e^x}{2}$$

Now just insert $x = 0$:

$$\lim_{x\to 0} \frac{9e^{3x} - e^x}{2} = 4$$

Example 9–3: Evaluate the limit $\lim_{x\to\infty} \frac{e^x}{x^3}$.

Solution: Indeterminacy of the form $\frac{\infty}{\infty}$ $\Rightarrow$ use L'Hôpital.

$$\begin{aligned} \lim_{x\to\infty} \frac{e^x}{x^3} &= \lim_{x\to\infty} \frac{e^x}{3x^2} \\ &= \lim_{x\to\infty} \frac{e^x}{6x} \\ &= \lim_{x\to\infty} \frac{e^x}{6} \\ &= \infty \end{aligned}$$

The result would be the same if it were x^{30} rather than x^3. Exponential function increases faster than all polynomials.

Example 9–4: Evaluate the limit $\lim_{x\to\infty} \frac{\ln x}{x^2}$.

Solution: Indeterminacy of the form $\frac{\infty}{\infty}$ $\Rightarrow$ use L'Hôpital.

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln x}{x^2} &= \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} \\ &= \lim_{x\to\infty} \frac{1}{2x^2} \\ &= 0 \end{aligned}$$

The result would be 0 for any x^k. Logarithmic function increases slower than all polynomials.

Example 9–5: Evaluate the limit $\lim\limits_{x\to 1} \dfrac{x^{10}-1}{x^7-1}$.

Solution: It is possible to solve this question using the algebraic identities:

$$x^{10}-1 = \big(x-1\big)\big(x^9+x^8+\cdots+x+1\big)$$

$$x^7-1 = \big(x-1\big)\big(x^6+x^5+\cdots+x+1\big)$$

$$\begin{aligned}\lim_{x\to 1} \frac{x^{10}-1}{x^7-1} &= \lim_{x\to 1} \frac{x^9+x^8+\cdots+x+1}{x^6+x^5+\cdots+x+1} \\ &= \frac{10}{7}\end{aligned}$$

but this is too complicated. Limit is in the form $\dfrac{0}{0}$ and using L'Hôpital gives the same result easily.

$$\begin{aligned}\lim_{x\to 1} \frac{x^{10}-1}{x^7-1} &= \lim_{x\to 1} \frac{10x^9}{7x^6} \\ &= \frac{10}{7}\end{aligned}$$

Example 9–6: Evaluate the limit $\lim\limits_{x\to 0} \dfrac{(1+x)^{4/3}-1}{x}$.

Solution: Indeterminacy of the form $\dfrac{0}{0}$ $\Rightarrow$ Use L'Hôpital:

$$\begin{aligned}\lim_{x\to 0} \frac{(1+x)^{4/3}-1}{x} &= \lim_{x\to 0} \frac{\frac{4}{3}(1+x)^{1/3}}{1} \\ &= \frac{4}{3}\end{aligned}$$

Example 9–7: Evaluate the following limit: (if it exists.)

$$\lim_{x\to 0} \frac{e^x-1-x-\frac{x^2}{2}}{x^4}$$

Solution: This limit is in the form $\dfrac{0}{0}$, so using L'Hôpital's rule we obtain:

$$\begin{aligned}\lim_{x\to 0} \frac{e^x-1-x-\frac{x^2}{2}}{x^4} &= \lim_{x\to 0} \frac{e^x-1-x}{4x^3} \\ &= \lim_{x\to 0} \frac{e^x-1}{12x^2} \\ &= \lim_{x\to 0} \frac{e^x}{24x}\end{aligned}$$

At this point, the limit is NOT in the form $\dfrac{0}{0}$, so we can NOT use L'Hôpital. Checking the numerator and denominator, we see that:

$$= \infty$$

Example 9–8: Evaluate the limit $\lim\limits_{x\to\infty} \dfrac{\ln x + x^2}{xe^x}$.

Solution: Indeterminacy of the form $\dfrac{\infty}{\infty}$ $\Rightarrow$ Use L'Hôpital:

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln x + x^2}{xe^x} &= \lim_{x\to\infty} \frac{\frac{1}{x}+2x}{e^x+xe^x} \\ &= \lim_{x\to\infty} \frac{-\frac{1}{x^2}+2}{e^x+e^x+xe^x} \\ &= 0\end{aligned}$$

EXERCISES

Evaluate the following limits (if they exist):

9–1) $\lim\limits_{x\to 0} \dfrac{e^{5x} - e^{4x} - x}{x^2}$

9–2) $\lim\limits_{x\to 0} \dfrac{e^{3x} - 1}{\ln(x+1)}$

9–3) $\lim\limits_{x\to 0} \dfrac{e^{x} - 1}{e^{2x} + 3x - 1}$

9–4) $\lim\limits_{x\to \infty} \dfrac{3x^2 + 4\ln x}{6x^2 + 7\ln x}$

9–5) $\lim\limits_{x\to \infty} \dfrac{2e^x + 5x}{7e^x + 8x + 12}$

9–6) $\lim\limits_{x\to \infty} \dfrac{\ln(x + x^4)}{x}$

9–7) $\lim\limits_{x\to 64} \dfrac{x^{1/3} - 4}{x^{1/2} - 8}$

9–8) $\lim\limits_{x\to 0} \dfrac{\sqrt{9+2x} - 3}{\sqrt{16+x} - 4}$

Evaluate the following limits (if they exist):

9–9) $\lim\limits_{x\to 2} \dfrac{x^3 - 2x^2 + 6x - 12}{x^3 - 2x^2 + 8x - 16}$

9–10) $\lim\limits_{x\to e} \dfrac{\ln x - 1}{x - e}$

9–11) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x} - 1}{x}$

9–12) $\lim\limits_{x\to 3} \dfrac{x^3 - 4x - 15}{x^2 + x - 12}$

9–13) $\lim\limits_{x\to \infty} \dfrac{\ln x}{\sqrt[3]{x}}$

9–14) $\lim\limits_{x\to 1} \dfrac{x^6 - 1}{x^4 - 1}$

9–15) $\lim\limits_{x\to 3} \dfrac{e^x - e^3}{x^2 - 9}$

9–16) $\lim\limits_{x\to 0} \dfrac{4^x - 1}{2^x - 1}$

Evaluate the following limits (if they exist):

9–17) $\lim_{x \to 2} \frac{\ln \frac{x}{2}}{x(x-2)}$

9–18) $\lim_{x \to 0} \frac{\sqrt{a+bx} - \sqrt{a+cx}}{x}$

9–19) $\lim_{x \to 0} \frac{(1+x)^k - 1 - kx}{x^2}$

9–20) $\lim_{x \to 1} \frac{\ln x}{x-1}$

9–21) $\lim_{x \to 1} \frac{\ln x - x + 1}{(x-1)^2}$

9–22) $\lim_{x \to 1/2} \frac{\ln(2x)}{2x^2 + x - 1}$

9–23) $\lim_{x \to \infty} x^3 e^{-x}$

9–24) $\lim_{x \to 0^+} x \ln x$

ANSWERS

9–1) $\frac{9}{2}$

9–2) 3

9–3) $\frac{1}{5}$

9–4) $\frac{1}{2}$

9–5) $\frac{2}{7}$

9–6) 0

9–7) $\frac{1}{3}$

9–8) $\frac{8}{3}$

9–9) $\frac{5}{6}$

9–10) $\frac{1}{e}$

9–11) $\frac{1}{2}$

9–12) $\frac{23}{7}$

9–13) 0

9–14) $\frac{3}{2}$

9–15) $\frac{e^3}{6}$

9–16) 2

9–17) $\frac{1}{4}$

9–18) $\frac{b-c}{2\sqrt{a}}$

9–19) $\frac{k(k-1)}{2}$

9–20) 1

9–21) $-\frac{1}{2}$

9–22) $\frac{2}{3}$

9–23) 0

9–24) 0

Chapter 10

Finding Maximum and Minimum Values

Local and Absolute Extrema:
Extremum is either minimum or maximum. Extrema is the plural form.

Absolute Extrema: If

$$f(c) \leqslant f(x)$$

for all x on a set S of real numbers, $f(c)$ is the absolute minimum value of f on S.

Similarly if

$$f(c) \geqslant f(x)$$

for all x on S, $f(c)$ is the absolute maximum value of f on S.

Local Extrema: $f(c)$ is local minimum if

$$f(c) \leqslant f(x)$$

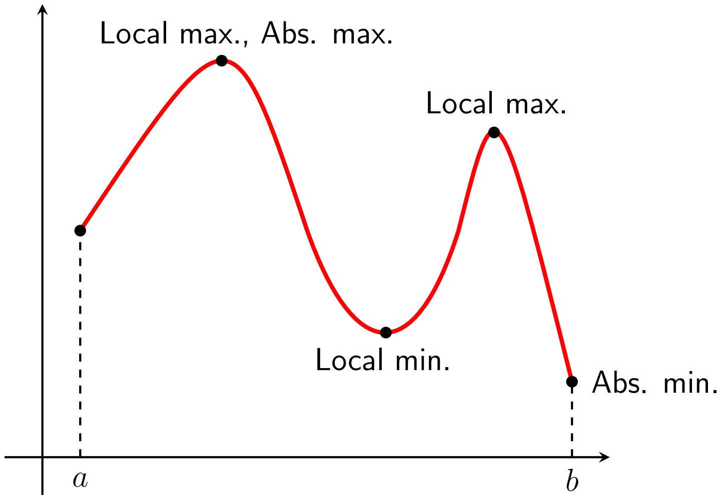
for all x in some open interval containing c.

Similarly, $f(c)$ is local maximum if

$$f(c) \geqslant f(x)$$

for all x in some open interval containing c.

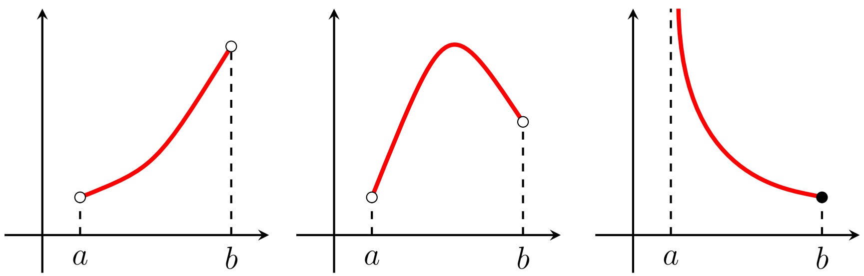
Local extrema are points that are higher (or lower) than the points around them.



As you can see in the figure, a point can be both local and absolute extremum. Also, it may be an absolute extremum without being a local one or vice versa.

Question: Does a continuous function always have an absolute maximum and an absolute minimum value?

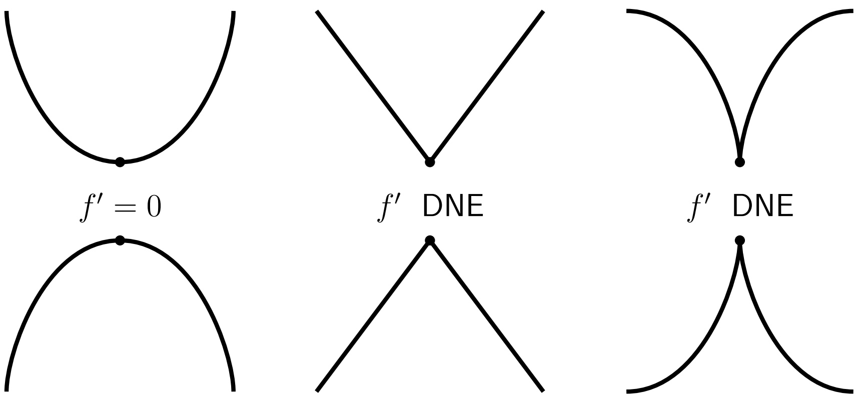
This depends on the interval. It may or may not have such values on an open interval.



No max. or min. Max. but no min. Min. but no max.

Theorem: If the function f is continuous on the closed interval $[a, b]$, then f has a maximum and a minimum value on $[a, b]$.

Critical Point: A number c is called a critical point of the function f if $f'(c) = 0$ or $f'(c)$ does not exist.



The main ideas about extremum points can be summarized as:

1. f can have local extremum only at a critical point.
2. f can have absolute extremum only at a critical point or an endpoint.

For example, the local extremum point of the parabola

$$f(x) = ax^2 + bx + c$$

will be at the point $x = -\dfrac{b}{2a}$ (called the vertex) because this is the point where the derivative is zero:

$$f'(x) = 2ax + b = 0 \quad \Rightarrow \quad x = -\frac{b}{2a}$$

If $a > 0$ it is a minimum and if $a < 0$ it is a maximum.

The local minimum point of the function $f(x) = |ax + b|$ will be at the point $x = -\dfrac{b}{a}$ because this is the point where the derivative is undefined.

How to find absolute extrema:

- Find the points where $f' = 0$.
- Find the points where f' does not exist.
- Consider such points only if they are **inside** the given interval.
- Consider endpoints.
- Check all candidates. Both absolute minimum and maximum are among them.

Example 10–1: Find the maximum and minimum values of

$$f(x) = -x^2 + 10x + 2$$

on the interval $[1, 4]$.

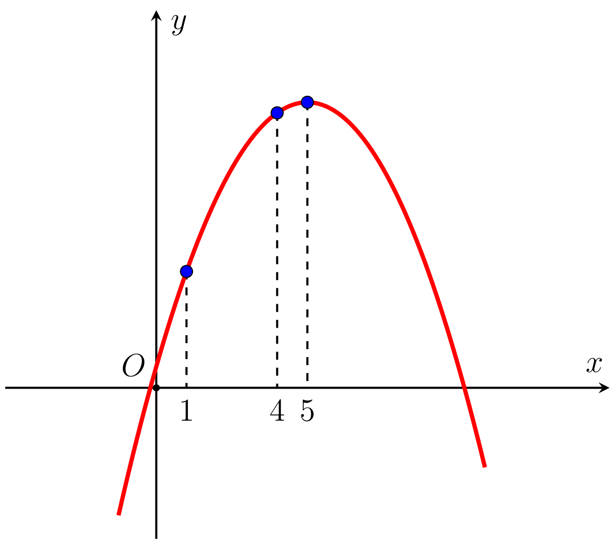
Solution: Let's find the critical points first:

$$f' = -2x + 10 = 0$$

$\Rightarrow \quad x = 5$ is the only critical point. But it is not in our interval $[0, 4]$, so our candidates for extrema are the endpoints:

x	$f(x)$
1	11
4	26

Clearly, absolute minimum is 11 and it occurs at $x = 1$. Absolute maximum is 26 and it occurs at $x = 4$.



Example 10–2: Find the maximum and minimum values of

$$f(x) = -x^2 + 10x + 2$$

on the interval $[2, 10]$.

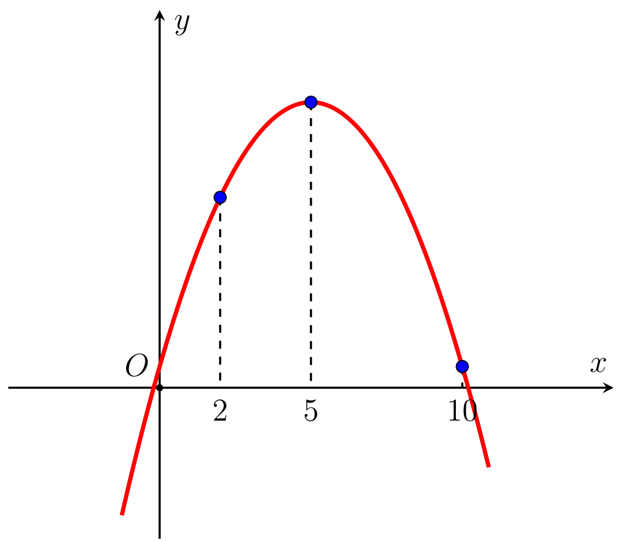
Solution: Although it is the same function, interval is different.

$$f'(x) = -2x + 10 = 0$$

$\Rightarrow \quad x = 5$ is the only critical point. It is inside the interval.

x	$f(x)$
2	18
5	27
10	2

Absolute minimum is 2 and it occurs at $x = 10$. Absolute maximum is 27 and it occurs at $x = 5$.



Example 10–3: Find the maximum and minimum values of

$$f(x) = |x - 8|$$

on the interval $[6,\ 12]$.

Solution: Let's write the function in piecewise defined form:

$$f(x) = \begin{cases} -x+8 & \text{if} \quad x < 8 \\ x-8 & \text{if} \quad x \geqslant 8 \end{cases}$$

Derivative is:

$$f'(x) = \begin{cases} -1 & \text{if} \quad x < 8 \\ 1 & \text{if} \quad x > 8 \end{cases}$$

Derivative is never zero. The only critical point is $x = 8$. Derivative does not exist at that point.

Now we need a table that shows all critical points in the interval and endpoints:

x	$f(x)$
6	2
8	0
12	4

We can see that absolute minimum is 0 and absolute maximum is 4.

Example 10–4: Find the maximum and minimum values of

$$f(x) = \left|16 - x^2\right|$$

on the interval $[-3,\ 5]$.

Solution: First, express f as a piecewise defined function:

$$f(x) = \begin{cases} x^2-16 & \text{if} \quad x < -4 \\ 16-x^2 & \text{if} \quad -4 \leqslant x \leqslant 4 \\ x^2-16 & \text{if} \quad x > 4 \end{cases}$$

Derivative is:

$$f'(x) = \begin{cases} 2x & \text{if} \quad x < -4 \\ -2x & \text{if} \quad -4 < x < 4 \\ 2x & \text{if} \quad x > 4 \end{cases}$$

f' is zero at $x = 0$ and it is undefined at $x = \pm 4$. We will not consider $x = -4$ because it is outside the interval. So, critical points in the interval are:

$$x = 0, \quad x = 4$$

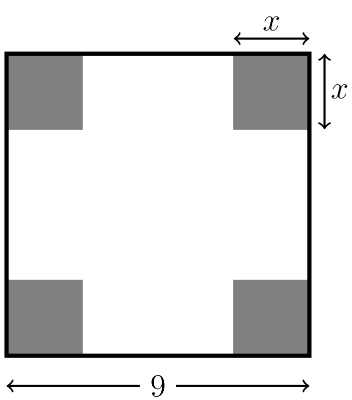
Together with the endpoints, we can make the following table:

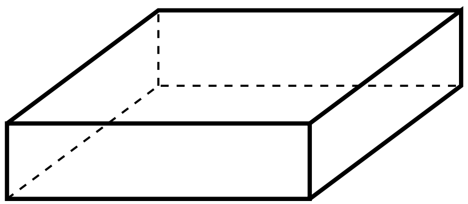
x	$f(x)$	
-3	7	
0	16	Abs. Max.
4	0	Abs. Min.
5	9	

Applied Optimization:
Finding the maximum or minimum of a function has many real-life applications. For these problems:

- Express the quantity to be maximized or minimized as a function of the independent variable. (We will call it x)
- Determine the interval over which x changes.
- Solve the problem in the usual way. (Find the critical points, check the function at critical points and endpoints)

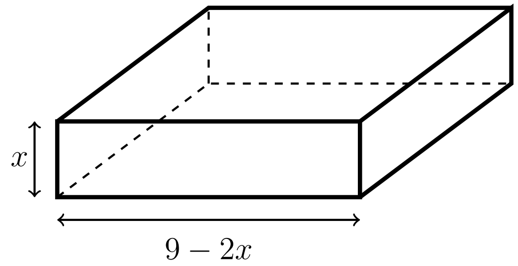
Example 10–5: A piece of cardboard is shaped as a 9×9 square. We will cut four small squares from the corners and make an open top box. What is the maximum possible volume of the box?





Solution: If the squares have edge length x, we can express the volume as:

$$V(x) = x(9-2x)^2 = 81x - 36x^2 + 4x^3$$



Considering the maximum and minimum possible values, we can see that $x \in \left[0, \frac{9}{2}\right]$. Now we can use maximization procedure:

$$V'(x) = 81 - 72x + 12x^2 = 0$$

$$27 - 24x + 4x^2 = 0$$

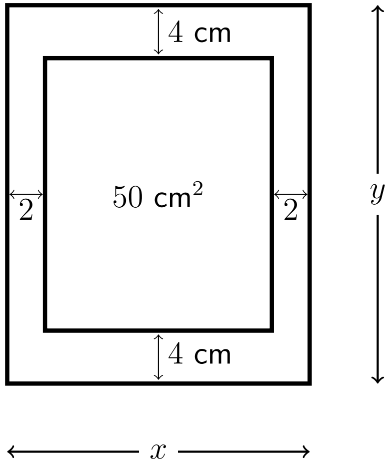
$$(2x-9)(2x-3) = 0$$

$$x = \frac{9}{2} \quad \text{or} \quad x = \frac{3}{2}$$

Checking all critical and endpoints, we find that $x = \dfrac{3}{2}$ gives the maximum volume, which is:

$$V = 54.$$

Example 10–6: You are designing a rectangular poster to contain 50 cm^2 of picture area with a 4 cm margin at the top and bottom and a 2 cm margin at each side. Find the dimensions x and y that will minimize the total area of the poster.



Solution: $(x-4)(y-8) = 50 \quad \Rightarrow \quad y = \dfrac{50}{x-4} + 8$

$$A = xy = x\left(\frac{50}{x-4} + 8\right)$$

$$A' = \frac{50}{x-4} + 8 - \frac{50x}{(x-4)^2} = 0$$

$$\frac{200}{(x-4)^2} = 8 \quad \Rightarrow \quad (x-4)^2 = 25$$

$$\Rightarrow \quad x = 9 \quad \text{and} \quad y = 18.$$

Example 10–7: You are selling tickets for a concert. If the price of a ticket is $\$15$, you expect to sell 600 tickets. Market research reveals that, sales will increase by 40 for each $\$0.5$ price decrease, and decrease by 40 for each $\$0.5$ price increase. For example, at $\$14.5$ you will sell 640 tickets. At $\$16$ you will sell 520 tickets.

What should the ticket price be for largest possible revenue?

Solution: We need to define our terms first:

- x denotes the sale price of a ticket in \$,
- N denotes the number of tickets sold,
- R denotes the revenue.

According to market research, $N = 600 + 40\,\dfrac{15-x}{0.5}$.
In other words:

$N = 600 + 80\big(15 - x\big) = 1800 - 80x.$

Note that we sell zero tickets if $x = \dfrac{1800}{80} = 22.5.$
(That's the highest possible price.)

Revenue is:
$$\begin{aligned} R &= Nx \\ &= (1800 - 80x)x \\ &= 1800x - 80x^2 \end{aligned}$$

This is a maximization problem where the interval of the variable is: $x \in \big[0,\, 22.5\big]$.

$R' = 1800 - 160x = 0 \quad \Rightarrow \quad x = 11.25$

Checking the critical point $x = 11.25$ and endpoints 0 and 22.5 we see that the maximum revenue occurs at $x = 11.25$.

Example 10–8: A helicopter will cover a distance of 235 km. with constant speed v km/h. The amount of fuel used during flight in terms of liters per hour is

$$75 + \frac{v}{3} + \frac{v^2}{1200}.$$

Find the speed v that minimizes total fuel used during flight.

Solution: The time it takes for flight is: $t = \dfrac{235}{v}$

The total amount of fuel consumed is:

$$\left(75 + \frac{v}{3} + \frac{v^2}{1200}\right) \cdot t = 235 \cdot \left(\frac{75}{v} + \frac{1}{3} + \frac{v}{1200}\right)$$

In other words we have to find v that minimizes $f(v)$ on $v \in \big(0, \infty\big)$ where:

$$f(v) = \frac{75}{v} + \frac{1}{3} + \frac{v}{1200}$$

Note that the distance 235 km. is not relevant. Once we find the optimum speed, it is optimum for all distances.

$$f'(v) = -\frac{75}{v^2} + \frac{1}{1200} = 0$$

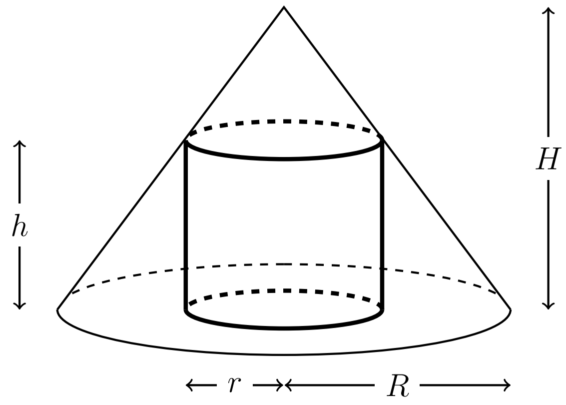
$$\Rightarrow \quad v^2 = 75 \cdot 1200 = 90\,000$$

$$\Rightarrow \quad v = 300$$

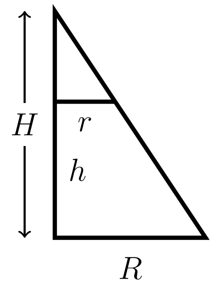
This value clearly gives the minimum, because:

$$\lim_{v \to 0} f = \lim_{v \to \infty} f = \infty.$$

Example 10–9: A cylinder is inscribed in a cone of radius R, height H. What is the maximum possible the volume of the cylinder?



Solution:



$$V = \pi r^2 h$$

Similar triangles: $\dfrac{H-h}{H} = \dfrac{r}{R}$

$$\Rightarrow \quad h = H\left(1 - \frac{r}{R}\right)$$

$$V = \pi r^2 H\left(1 - \frac{r}{R}\right) = \pi H\left(r^2 - \frac{r^3}{R}\right)$$

$$\frac{dV}{dr} = \pi H\left(2r - \frac{3r^2}{R}\right) = 0$$

$$2r = \frac{3r^2}{R} \quad \Rightarrow \quad r = \frac{2R}{3} \quad \Rightarrow \quad h = \frac{H}{3}$$

Maximum Volume: $V = \dfrac{4}{27}\pi R^2 H.$

EXERCISES

Find the absolute maximum and minimum values of $f(x)$ on the given interval:

10–1) $f(x) = x^{\frac{2}{3}}$ on $[-2, 3]$

10–2) $f(x) = 10x(2 - \ln x)$ on $[1, e^2]$

10–3) $f(x) = 12 - x^2$ on $[2, 4]$

10–4) $f(x) = 12 - x^2$ on $[-2, 4]$

10–5) $f(x) = 3x^3 - 16x$ on $[-2, 1]$

10–6) $f(x) = x + \dfrac{9}{x}$ on $[1, 4]$

10–7) $f(x) = 3x^5 - 5x^3$ on $[-2, 2]$

10–8) $f(x) = |3x - 5|$ on $[0, 2]$

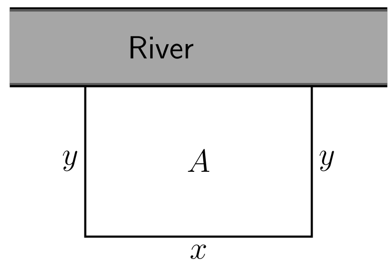
10–9) $f(x) = |x^2 + 6x - 7|$ on $[-8, 2]$

10–10) $f(x) = x\sqrt{1 - x^2}$ on $[-1, 1]$

10–11) $f(x) = e^{-x^2}$ on $[-1, 2]$

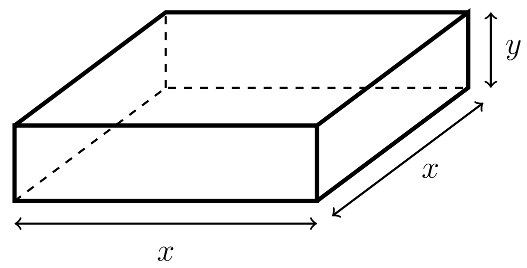
10–12) $f(x) = \dfrac{120}{\sqrt{x}}$ on $[16, 36]$

10–13) We will cover a rectangular area with a 36m-long fence. The area is near a river so we will only cover the three sides. Find the maximum possible area.



$$y + x + y = 36$$

10–14) An open top box has volume 75 cm^3 and is shaped as seen in the figure. Material for base costs 12\$/cm^2 and material for sides costs 10\$/cm^2. Find the dimensions x and y that give the minimum total cost.



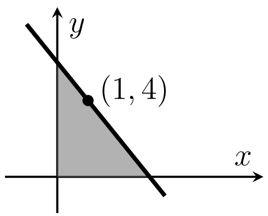
10–15) Find the dimensions of the right circular cylinder of the greatest volume if the surface area is 54π.

10–16) What is the maximum possible area of the rectangle with its base on the $x-$axis and its two upper vertices are on the graph of $y = 4 - x^2$?

10–17) Find the shortest distance between the point $(2, 0)$ and the curve $y = \sqrt{x}$.

10–18) Find the point on the line $y = ax + b$ that is closest to origin.

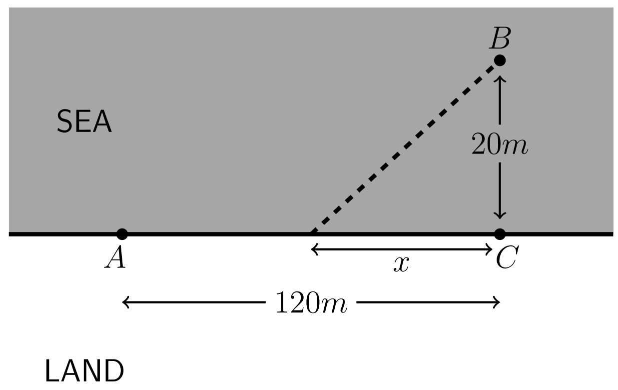
10–19) We choose a line passing through the point $(1, 4)$ and find the area in the first quadrant bounded by the line and the coordinate axes. What line makes this area minimum?



10–20) Two vertical poles are 21 meters apart. Their heights are 12m and 16m. A cable is stretched from the top of first pole to a point on the ground and then to the top of the second pole. Find the minimum possible length of the cable.

10–21) Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius R.

10–22) A swimmer is drowning on point B. You are at point A. You may run up to point C and then swim, or you may start swimming a distance x earlier. Assume your running speed is 5 m/s and your swimming speed is 3 m/s. What is the ideal x?



10–23) A coffee chain has 20 shops in a city. Average daily profit per shop is $\$3000$. Each new shop decreases the average profit of all shops by $\$100$. For example, if the company opens 3 new shops, average profit becomes $\$2700$.

What is the ideal number of shops, assuming the company wants to maximize total profit?

10–24) A $500-$room hotel's nightly rent is $\$80$ and it is full every night. For each $\$1$ increase in rent, 5 fewer rooms are rented. For example, if rent is $\$100$ there are 400 full and 100 empty rooms. The cost of service per room (for full rooms) is $\$40$ per day. What is the nightly rent that maximizes profit? What is the maximum profit?

ANSWERS

10–1) Absolute Minimum: 0, Absolute Maximum: $\sqrt[3]{9}$.

10–2) Absolute Minimum: 0, Absolute Maximum: $10e$.

10–3) Absolute Minimum: -4, Absolute Maximum: 8.

10–4) Absolute Minimum: -4, Absolute Maximum: 12.

10–5) Absolute Minimum: -13, Absolute Maximum: $\frac{128}{9}$.

10–6) Absolute Minimum: 6, Absolute Maximum: 10.

10–7) Absolute Minimum: -56, Absolute Maximum: 56.

10–8) Absolute Minimum: 0, Absolute Maximum: 5.

10–9) Absolute Minimum: 0, Absolute Maximum: 16.

10–10) Absolute Minimum: $-\frac{1}{2}$, Absolute Maximum: $\frac{1}{2}$.

10–11) Absolute Minimum: $\frac{1}{e^2}$, Absolute Maximum: 1.

10–12) Absolute Minimum: 20, Absolute Maximum: 30.

10–13) $A = 162$

10–14) $x = 5, \quad y = 3$

10–15) $r = 3, \quad h = 6$

10–16) $\frac{32}{3\sqrt{3}}$

10–17) $\frac{\sqrt{7}}{2}$

10–18) $\left(\frac{-ab}{1+a^2}, \frac{b}{1+a^2}\right)$

10–19) $y = -4x + 8$

10–20) $35m$

10–21) $r = \frac{\sqrt{2}}{\sqrt{3}}R, \quad h = \frac{2}{\sqrt{3}}R$

10–22) $x = 15m$

10–23) 25

10–24) $110, \quad 24\,500$

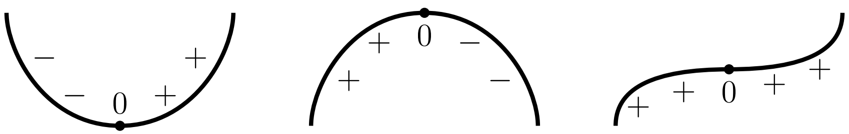
Chapter 11

Curve Sketching

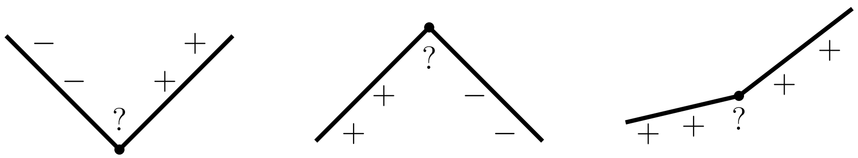
First Derivative Test: At a critical point, the derivative is zero or undefined. Let f be a continuous function and let $x = c$ be a critical point of it. Suppose f' exists in some interval containing c except possibly at c.

f has a local extremum at c if and only if f' changes sign at c.

- Sign change: $-$ to $+$ $\Rightarrow$ $f(c)$ is a local minimum.
- Sign change: $+$ to $-$ $\Rightarrow$ $f(c)$ is a local maximum.



Local Min. Local Max. Neither



Example 11–1: Find the intervals where $f(x) = 2x^3 - 9x^2 + 5$ is increasing and decreasing and local extrema of this function.

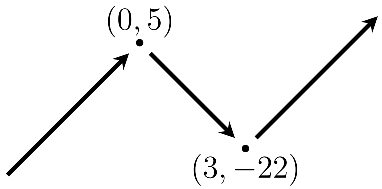
Solution: $f'(x) = 6x^2 - 18x = 0 \quad \Rightarrow \quad x = 0$ or $x = 3$.

There are two critical points, 0 and 3. Note that $f(0) = 5$ and $f(3) = -22$.

x changes sign at 0 and $(x-3)$ changes sign at 3. We can find the sign of $x(x-3)$ by multiplying these signs.

x		0		3	
$(x-3)$	$-$		$-$	0	$+$
$f' = x(x-3)$	$+$	0	$-$	0	$+$
f	increasing		decreasing		increasing

Based on this table, we can see that the graph is roughly like this:



Therefore $(0, 5)$ is local maximum and $(3, -22)$ is local minimum.

Concavity: The graph of a differentiable function is concave up if f' increasing, it is concave down if f' decreasing.

Test for Concavity:

- If $f''(x) > 0$, then f is concave up at x.
- If $f''(x) < 0$, then f is concave down at x.

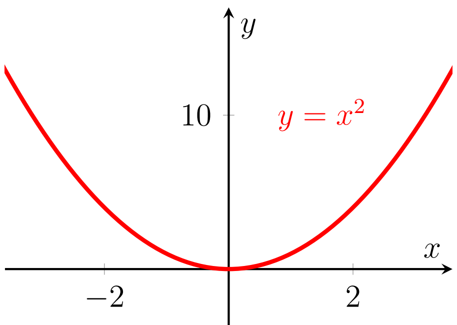
Inflection Point: An inflection point is a point where the concavity changes. In other words, if:

- f is continuous at $x = a$,
- $f'' > 0$ on the left of a and $f'' < 0$ on the right, or vice versa.

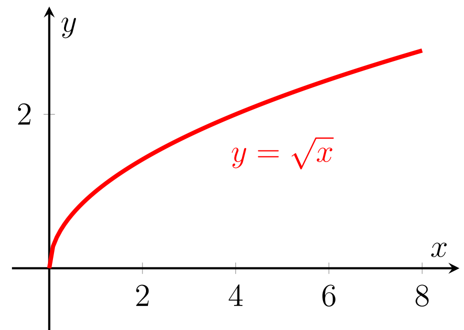
then $x = a$ is an inflection point.

This means either $f''(a) = 0$ or $f''(a)$ does not exist.

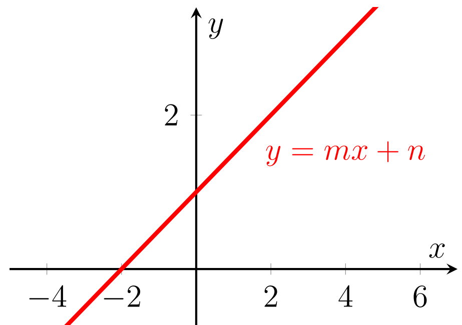
Examples:



Concave UP



Concave DOWN



NEITHER

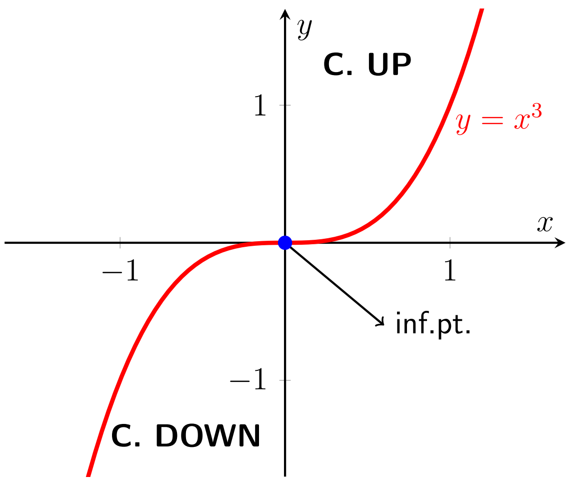
Example 11–2: Determine the concavity of $f(x) = x^3$. Find inflection points. (If there is any.)

Solution:

$$\begin{aligned} f &= x^3 \\ f' &= 3x^2 \\ f'' &= 6x \end{aligned}$$

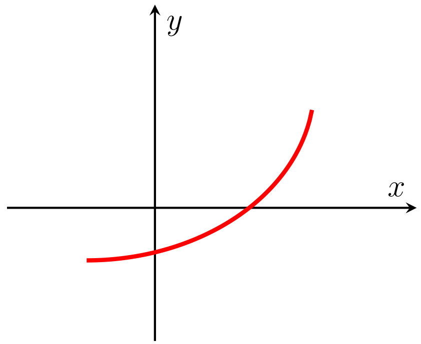
- For $x > 0$, $f'' > 0 \quad \Rightarrow \quad f$ is concave up.
- For $x < 0$, $f'' < 0 \quad \Rightarrow \quad f$ is concave down.
- $x = 0$ is the inflection point.

x		0	
f''	$-$	0	$+$
f is:	concave down		concave up



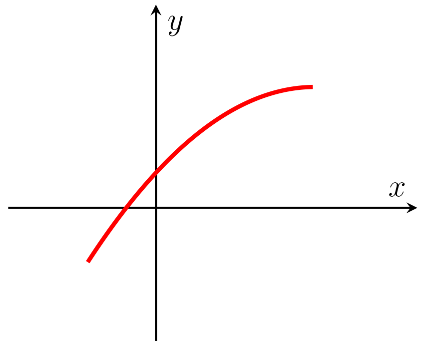
Shape of a graph based on first and second derivatives:

$f' > 0, \quad f'' > 0$



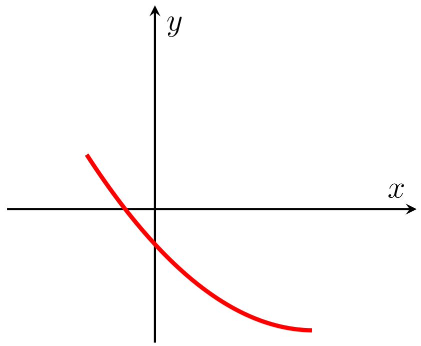
Increasing, Concave up.

$f' > 0, \quad f'' < 0$



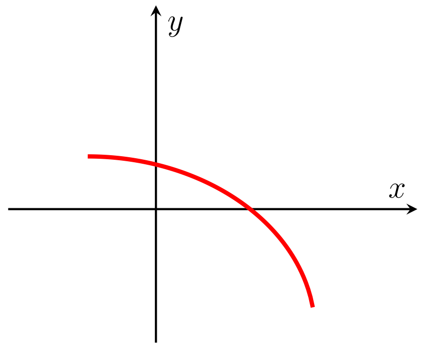
Increasing, Concave down.

$f' < 0, \quad f'' > 0$



Decreasing, Concave up.

$f' < 0, \quad f'' < 0$



Decreasing, Concave down.

Curve Sketching:

- Identify domain of f, symmetries, x and y intercepts. (if any)
- Find first and second derivatives of f.
- Find critical points, inflections points.
- Make a table and include all this information.
- Sketch the curve using the table.

Example 11–3: Sketch the graph of $f(x) = x^3 + 3x^2 - 24x$.

Solution: $\lim_{x\to\infty} f = +\infty, \quad \lim_{x\to-\infty} f = -\infty$

$$\begin{aligned} f' &= 3x^2 + 6x - 24 \\ &= 3(x+4)(x-2) \end{aligned}$$

$f' = 0 \quad \Rightarrow \quad x = -4,$ and $x = 2.$
These are the critical points.

$f'' = 6x + 6 = 0 \quad \Rightarrow \quad x = -1.$
This is the inflection point.

Some specific points on the graph are:

$f(-4) = 80, \quad f(-1) = 26.$

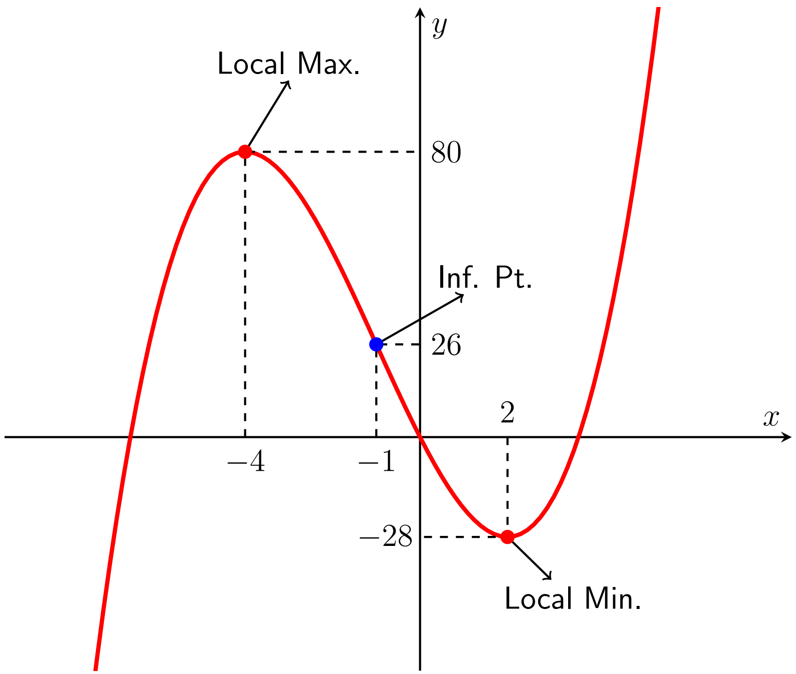
$f(0) = 0, \quad f(2) = -28.$

The equation $f(x) = 0$ gives $x = 0$ or $x^2 + 3x - 24 = 0$ in other words $x = \dfrac{-3 \pm \sqrt{105}}{2}$. Using a calculator we find $x_1 = -6.6, \ x_2 = 3.6$ but it is possible to sketch the graph without

these points. Putting all this information on a table, we obtain:

x		-4		-1		2	
f'	$+$	0	$-$		$-$	0	$+$
f''	$-$		$-$	0	$+$		$+$
f	↗		↘		↘		↗

Based on this table, we can sketch the graph as:



EXERCISES

Determine the intervals where the following functions are increasing and decreasing:

11–1) $f(x) = x^3 - 12x - 5$

11–2) $f(x) = 16 - 4x^2$

11–3) $f(x) = \dfrac{1}{(x-4)^2}$

11–4) $f(x) = \dfrac{x^2 - 3}{x - 2}$

11–5) $f(x) = 4x^5 + 5x^4 - 40x^3$

11–6) $f(x) = x^4 e^{-x}$

11–7) $f(x) = \dfrac{\ln x}{x}$

11–8) $f(x) = 5x^6 + 6x^5 - 45x^4$

11–9) $f(x) = x^4 - 2x^2 + 1$

11–10) $f(x) = \dfrac{x}{x+1}$

Identify local maxima, minima and inflection points, then sketch the graphs of the following functions:

11–11) $f(x) = x^3 - 3x^2 - 9x + 11$

11–12) $f(x) = -2x^3 + 21x^2 - 60x$

11–13) $f(x) = 3x^4 + 4x^3 - 36x^2$

11–14) $f(x) = (x-1)^2(x+2)^3$

11–15) $f(x) = x^6 - 6x^5$

11–16) $f(x) = x^3 e^{-x}$

11–17) $f(x) = e^{-x^2}$

11–18) $f(x) = \dfrac{x}{x^2 + 1}$

11–19) $f(x) = x \ln |x|$

11–20) $f(x) = -x^4 + 32x^2$

ANSWERS

11–1) Increasing on $(-\infty,\ -2)$, decreasing on $(-2,\ 2)$, increasing on $(2,\ \infty)$.

11–2) Increasing on $(-\infty,\ 0)$, decreasing on $(0,\ \infty)$.

11–3) Increasing on $(-\infty,\ 4)$, decreasing on $(4,\ \infty)$.

11–4) Increasing on $(-\infty,\ 1)$, decreasing on $(1,\ 2) \cup (2,\ 3)$, increasing on $(3,\ \infty)$.

11–5) Increasing on $(-\infty,\ -3)$, decreasing on $(-3,\ 0) \cup (0,\ 2)$, increasing on $(2,\ \infty)$.

11–6) Decreasing on $(-\infty,\ 0)$, increasing on $(0,\ 4)$, decreasing on $(4,\ \infty)$.

11–7) Increasing on $(0,\ e)$, decreasing on $(e,\ \infty)$.

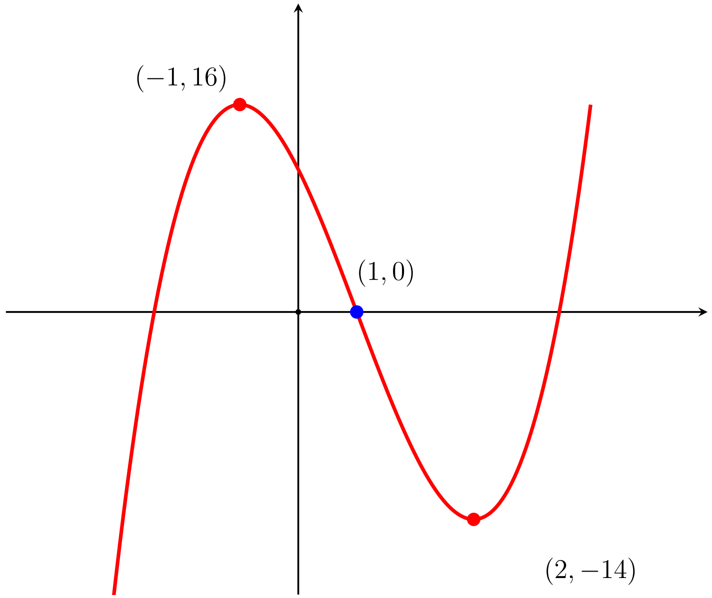
11–8) Decreasing on $(-\infty,\ -3)$, increasing on $(-3,\ 0)$, decreasing on $(0,\ 2)$, increasing on $(2,\ \infty)$.

11–9) Decreasing on $(-\infty,\ -1)$, increasing on $(-1,\ 0)$, decreasing on $(0,\ 1)$, increasing on $(1,\ \infty)$.

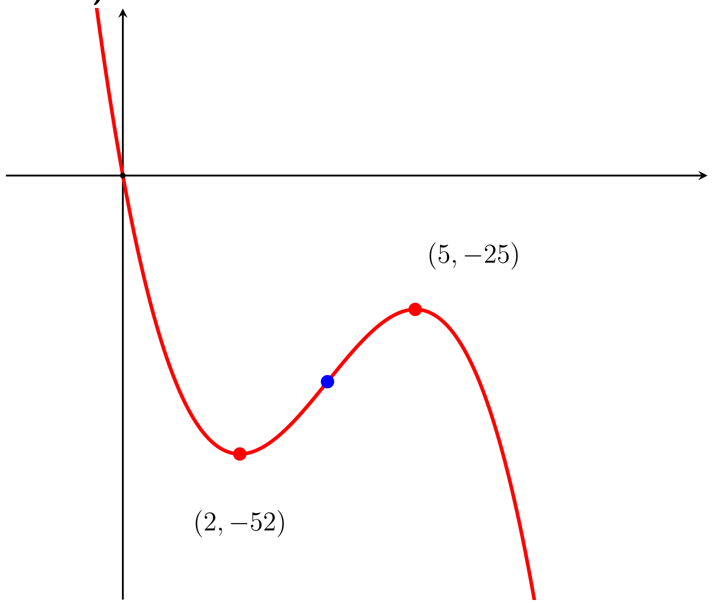
11–10) Increasing on $(-\infty,\ -1) \cup (-1,\ \infty)$.

(Blue dots denote inflection points, red dots local extrema.)

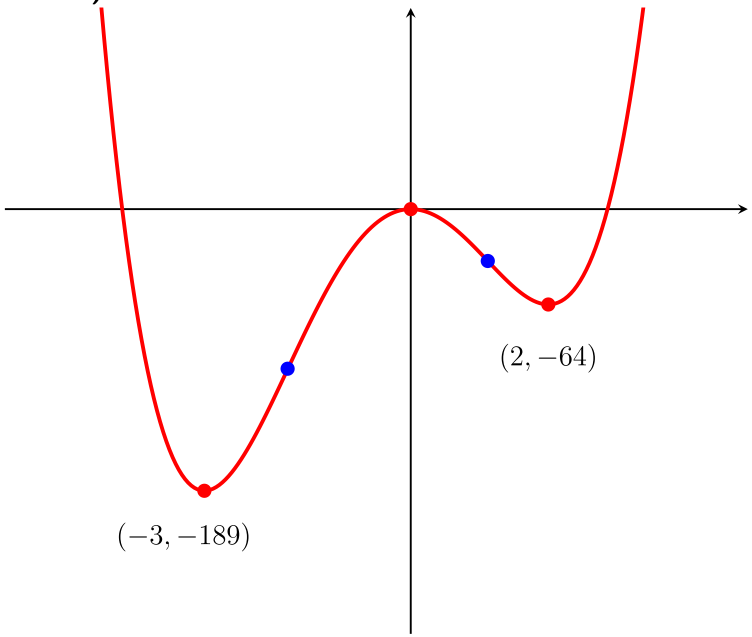
11–11)



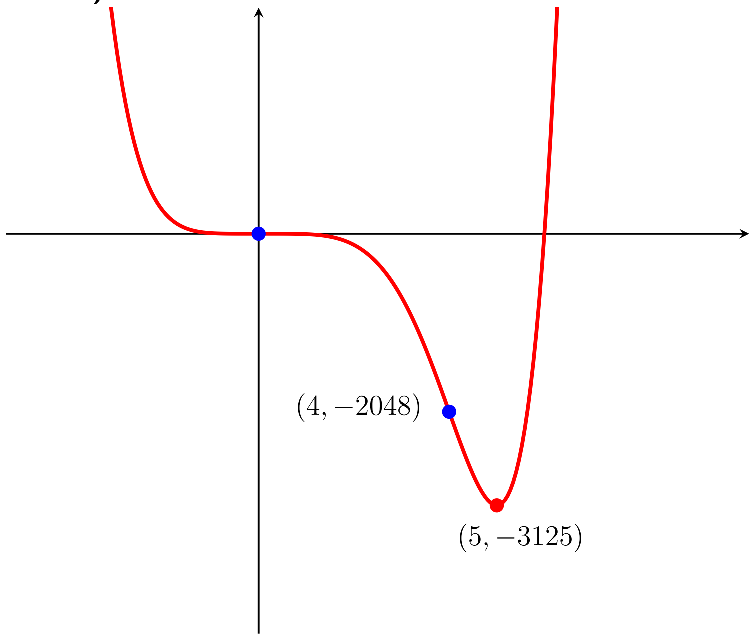
11–12)



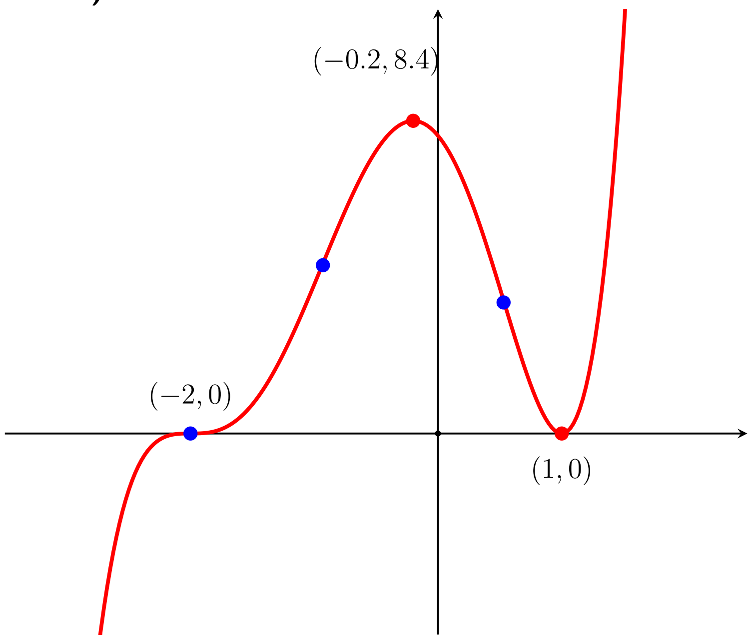
11–13)



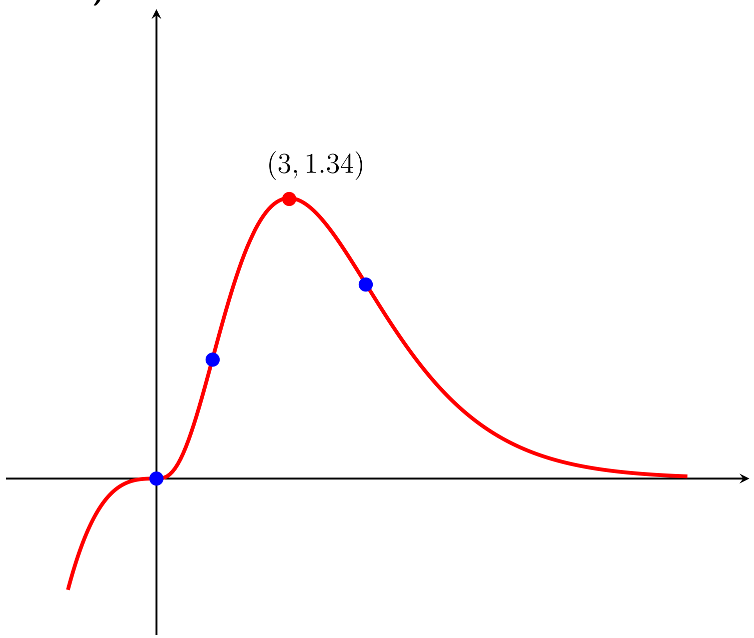
11–15)



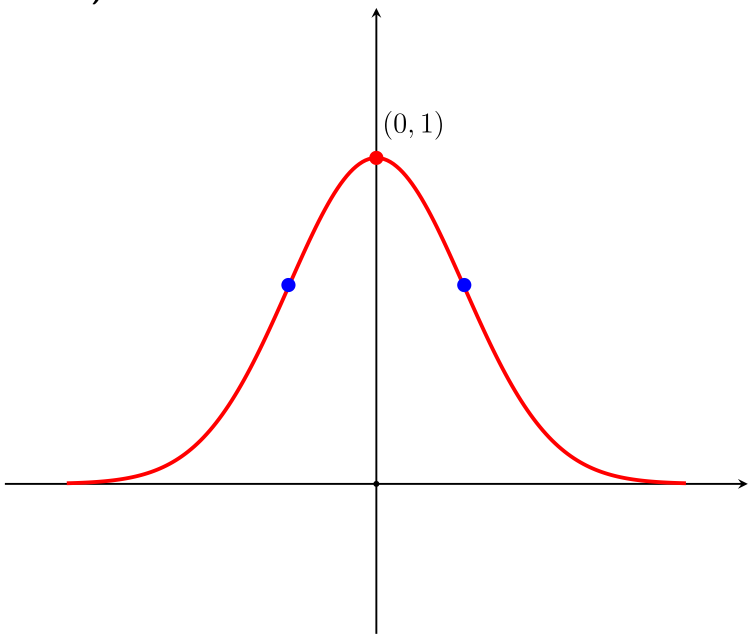
11–14)



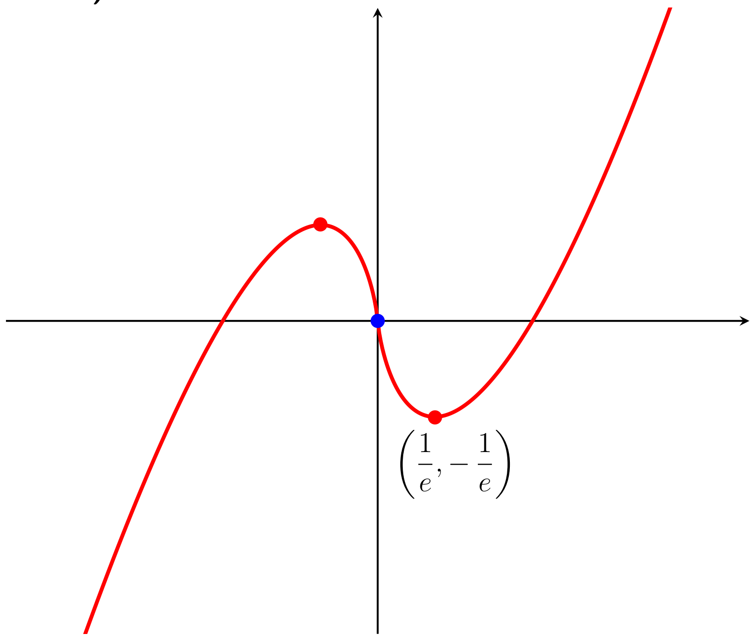
11–16)



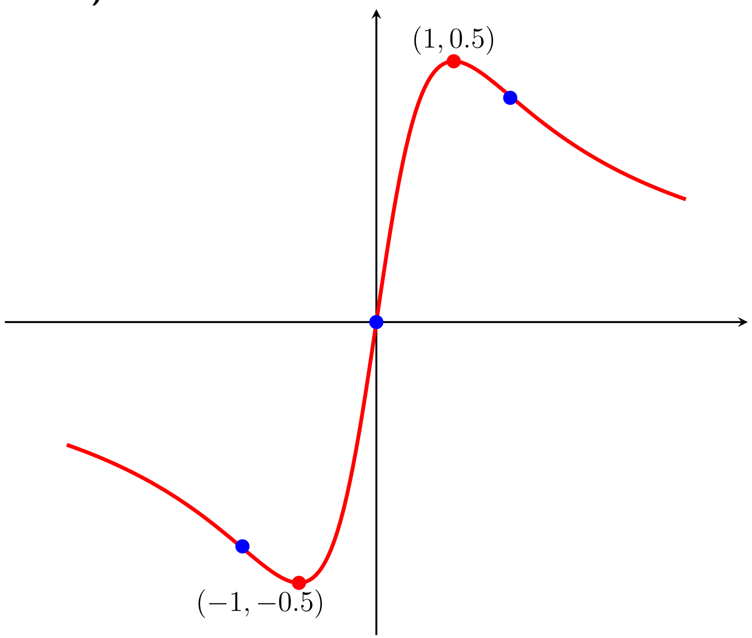
11–17)



11–19)



11–18)



11–20)