ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 105 - Business Mathematics I

2018-2019 Fall

FINAL EXAMINATION

14.~~08~~.01.2019, 15:00

-SOLUTIONS-

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 90 minutes

Question	Grade	Out of
1		20
2		20
3		15
4		15
5		15
6		21
Total		106

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. Evaluate the following limits if they exist. Show your work. Correct results with no work will <u>not get any points</u>.

a) (10 points) $\lim_{x\to\infty} \frac{e^x}{x^2+x} = \frac{\infty}{\infty}$

L'Hôspital's rule $\Rightarrow$ $\stackrel{\rightarrow}{=} \lim_{x\to\infty} \frac{e^x}{2x+1} \underset{\downarrow}{=} \lim_{x\to\infty} \frac{e^x}{2} = \boxed{\infty}$

$\frac{\infty}{\infty} \Rightarrow$ L.H.

b) (10 points) $\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = \frac{\infty}{\infty}$

$\underset{\text{L.H.}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \left(\frac{1}{x}\right)\cdot\left(2\sqrt{x}\right)$

$= \lim_{x\to\infty} \left(\frac{2}{\sqrt{x}}\right) = \frac{2}{\infty} = \boxed{0}$

2. a) (10 points) If $y = x^{x^2+1}$, find y' using logarithmic differentiation.

$$y = x^{x^2+1} \Rightarrow \ln y = (x^2+1)(\ln x) \xrightarrow{\frac{d}{dx}} \frac{y'}{y} = (2x)(\ln x) + (x^2+1)\left(\frac{1}{x}\right)$$

$$\Rightarrow y' = y\left[(2x)(\ln x) + x + \frac{1}{x}\right]$$

$$\boxed{y' = x^{x^2+1}\left[(2x)(\ln x) + x + \frac{1}{x}\right]}$$

b) (10 points) If $x + xy + y^2 = 7$ then find y' at the point where $x = 1$ using implicit differentiation. $(y>0)$

$$x=1 \Rightarrow 1 + y + y^2 = 7 \Rightarrow y^2 + y - 6 = 0 \Rightarrow (y+3)(y-2) = 0$$

$\cancel{y=-3}$ or $y=2$

$(y>0)$

point: $(1,2)$

$$1 + [y + x \cdot y'] + 2y \cdot y' = 0$$

$$y'[x+2y] = -y-1$$

$$y' = -\frac{y+1}{x+2y} \text{ at } (1,2) \Rightarrow$$

$$y'\Big|_{(1,2)} = -\frac{2+1}{1+2(2)} = \boxed{-\frac{3}{5}}$$

3. (15 points) Find the equation of the tangent line to the graph of $y = x\left(2^{2-x^2}\right)$ at the point $(1,2)$.

$y=(x)(2^{2-x^2}) \Rightarrow y' = 1.2^{2-x^2} + x\cdot(2^{2-x^2})(-2x)(\ln 2)$

$= 2^{2-x^2}\left[1-2x^2\ln 2\right]$

slope of tangent line $= m = y'\Big|_{(1,2)} = 2^{2-1^2}\cdot\left[1-2(1)^2\ln 2\right]$

$= \boxed{2[1-2\ln 2]}$

tangent line eqn.: $y-2 = [2(1-2\ln 2)](x-1)$

$y = 2(1-2\ln 2)(x) - \cancel{2} + 2\ln 2 + \cancel{2}$

$\boxed{y = 2(1-2\ln 2)(x) + 2\ln 2}$

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4. (15 points) Find real numbers a and b that make $f(x)$ continuous on $\mathbb{R}$ where

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{if } x < 2, \\ ax^2 - bx + 3, & \text{if } 2 \le x \le 3, \\ 2x - a + b, & \text{if } 3 < x. \end{cases}$$

Explain your answer in detail.

$\lim\limits_{\substack{x\to 2^-\\(x<2)}} \left(\dfrac{x^2-4}{x-2}\right) = \lim\limits_{x\to 2^-} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = 2+2=4$

$\lim\limits_{\substack{x\to 2^+\\(x>2)}} (ax^2-bx+3) = a(2)^2 - b(2) + 3 = 4a-2b+3$

for continuity at $x=2$:

$4a-2b+3=4$

$\Rightarrow \boxed{4a-2b=1}$

$\lim\limits_{\substack{x\to 3^-\\(x<3)}} (ax^2-bx+3) = a(9)-3b+3 = 9a-3b+3$

$\lim\limits_{\substack{x\to 3^+\\(x>3)}} (2x-a+b) = 2(3)-a+b = 6-a+b$

for continuity at $x=3$:

$9a-3b+3 = 6-a+b$

$\Rightarrow \boxed{10a-4b=3}$

$\left.\begin{matrix} 4a-2b=1 \\ 10a-4b=3 \end{matrix}\right\} \Rightarrow \boxed{a=\frac{1}{2}\ ,\ b=\frac{1}{2}}$

5. (15 points) Find two positive numbers x and y such that $2x + y = 6$ and x^2y is maximum. Show your work and conclude about the result using differentiation methods.

$$2x+y=6 \Rightarrow y=6-2x$$

$$f(x)=x^2(6-2x)=6x^2-2x^3$$

$$f'(x)=12x-6x^2=6x(2-x)=0 \Rightarrow \cancel{x=0},\ \boxed{x=2}$$

$x>0$

x values	0	2	
f'	$+$		$-$
f	↗		↘

max. value

$$x=2 \Rightarrow y=6-2(2)=6-4=2$$

So for $\boxed{x=2,\ y=2} \Rightarrow x^2y$ is maximum with value $(2)^2\cdot 2 = 8$

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6. Given that $f(x) = x^3 + 3x^2 - 24x$.

a) (3 points) Find the domain, x-intercept(s) and y-intercept(s) of $f(x)$ (If any).

$D_f=\mathbb{R}$

(x values of f)

$f(x)=x(\underbrace{x^2+3x-24}_{=0})=0 \Rightarrow$ x-intercepts: $\boxed{x=0 \ \& \ x=\frac{-3\mp\sqrt{105}}{2}}$

$$\Rightarrow x=\frac{-3\mp\sqrt{9-4(1)(-24)}}{2}=\frac{-3\mp\sqrt{105}}{2}$$

$x=0 \Rightarrow y=0 \Rightarrow$ y-intercept: $(0,0)$

b) (3 points) Find $f'(x)$ and the critical values (if they exist).

$$f'(x)=3x^2+6x-24=3(x^2+2x-8)=0$$

$$3(x+4)(x-2)=0 \Rightarrow x=-4,\ x=2$$

critical values: $\boxed{x=2 \ \& \ x=-4}$

c) (3 points) Find the intervals where $f(x)$ is increasing and decreasing.

	$-\infty$		-4		2		∞
$3(x^2+2x-8)=f'$		$+$	0	$-$	0	$+$	
f		$\nearrow$	max	$\searrow$	min.	$\nearrow$	

f is increasing on: $(-\infty,-4)\cup(2,\infty)$

f is decreasing on: $(-4,2)$

d) (3 points) Find the relative extrema of the function $f(x)$.

$x=2 \Rightarrow f(2)=(2)^3+3(2)^2-24(2)=8+12-48=-28$

$x=-4 \Rightarrow f(4)=(-4)^3+3(-4)^2-24(-4)=-64+48+96=80$

$$\boxed{\begin{array}{l} f \text{ has } \underline{\text{relative maximum}} \text{ at } (-4,80) \\ f \text{ has } \underline{\text{relative minimum}} \text{ at } (2,-28) \end{array}}$$

e) (3 points) Find $f''(x)$ and the inflection points (if they exist).

$$\left.\begin{array}{l} f''(x)=6x+6=0 \Rightarrow x=-1 \\ f(-1)=(-1)^3+3(-1)^2-24(-1)=-1+3+24=26 \end{array}\right\} \text{inflection point of } f \text{ is at: } (-1,26)$$

f) (3 points) Find the intervals where $f(x)$ is concave up and concave down.

		-1	
f''	$-$	0	$+$
f	$\cap$		$\cup$

f is concave down on: $(-\infty,-1)$

f is concave up on: $(-1,\infty)$

g) (3 points) Sketch the graph of $f(x)$. Clearly indicate on the graph the points you found in the parts **d)** and **e)**.